3 1761 06704277 0

Edm. John Senkler
1836.

Digitized by the Internet Archive
in 2007 with funding from
Microsoft Corporation

http://www.archive.org/details/syllabusofalgebr00pearuoft

P362.5

A

SYLLABUS OF ALGEBRA;

CONTAINING,

IN A COMPENDIOUS FORM,

ALL THE THEOREMS AND PRINCIPLES OF THAT SCIENCE (WITH THEIR DEMONSTRATIONS) WHICH ARE OF ANY UTILITY

IN THE

MORE ADVANCED PARTS OF MATHEMATICS.

BY

HENRY PEARSON, B. A.

OF TRINITY COLLEGE, CAMBRIDGE.

CAMBRIDGE:

PRINTED BY AND FOR J. HALL;

WHITTAKER, TREACHER, AND CO., AND SIMPKIN AND CO., LONDON; AND THURNAM, CARLISLE.

1834.

PREFACE.

THE very favourable manner in which a work upon Trigonometry, on a plan similar to the one now presented to the reader, was received by the University, induced the Author of that work to make the present attempt.

The principles so ably enunciated by Mr. Peacock, in his Treatise on Algebra, have been adopted as the basis of the following pages, as being, in the Author's opinion, by far the most clear and satisfactory of any that have been advanced; and he only regrets that the narrow limits of an abridgement like the present, have not enabled him to do full justice to that excellent work.

Most of the approved writers on the Science of Algebra have been likewise consulted, and wherever fuller information may be acquired than is contained in this book, the reference has invariably been made in the text.

The present work contains what are usually called the *first* and *third* parts of Algebra; that is, the whole of the science except that part of it which relates to the Theory of Algebraical

Expressions: and it will scarcely be necessary to remind those who are acquainted with the Author's similar work on Trigonometry, that the following pages are not intended as an *Elementary Treatise*, to be placed in the hands of one studying Algebra for the first time; but merely as a work containing, in a compendious form, all the theorems and principles in the above divisions of the science, which are of any utility in the more advanced parts of Mathematics.

LONDON, *August*, 1834.

CONTENTS.

PART I.

DEFINITIONS AND PRINCIPLES.

SECTION I.

Page

Definitions and Principles of Arithmetical Algebra 1

SECTION II.

Definitions and Principles of Symbolical Algebra 5

Law of the Permanence of Equivalent forms 7

CHAPTER I.

ON THE FUNDAMENTAL OPERATIONS OF ALGEBRA.

Addition 9

Subtraction . . . . . *ib.*

Multiplication 10

To find the square of a Polynomial quantity 11

Division 12

To find a Rationalizing Multiplier of a Surd 15

CHAPTER II.

ON ALGEBRAIC FRACTIONS.

SECTION I.

On the Addition, Subtraction, &c. of Algebraic Fractions 16

SECTION II.

On the Reduction of Algebraic Fractions to their lowest terms 20

Greatest Common Measure . . . . . *ib.*

Least Common Multiple 23

CHAPTER III.

ON DECIMAL FRACTIONS.

Page
General explanation of Decimals 25
Conversion of Fractions into Decimals 26
Circulating Decimals 28
Addition, Multiplication, &c. of Decimals..... 29

CHAPTER IV.

ON THE EXTRACTION OF ROOTS.

Number of Digits in a Root, as compared with the square, cube, &c..... 31
Extraction of the Square Root 32
Extraction of the Cube Root 36

CHAPTER V.

ON EQUATIONS.

General Principles 39
Equations having two unknown quantities 41
Quadratic Equations 42
General principle of Elimination 43

CHAPTER VI.

On Inequalities 46

CHAPTER VII.

ON RATIOS, PROPORTION, AND VARIATION.

SECTION I.

On Ratios 48

SECTION II.

On Proportion 50

SECTION III.

On Variation 54

CHAPTER VIII.

ON PROGRESSIONS.

Arithmetic Progression 57
Geometric 58
Harmonic 60

CHAPTER IX.

ON PERMUTATIONS AND COMBINATIONS.

SECTION I.

The quantities considered being all dissimilar 62

SECTION II.

The quantities considered being not all dissimilar 68

CHAPTER X.

ON THE BINOMIAL THEOREM.

General Proof 71
Relations of the co-efficients 76

CHAPTER XI.

ON THE THEORY OF NUMBERS.

SECTION I.

On Surds 81
Extraction of the Roots of Surds 84

SECTION II.

On the different Scales of Notation 89
Conditions of Divisibility 90
Casting out the Nines 91
On the Properties of odd or even Numbers ..... *ib*

SECTION III.

On Prime Numbers 93
Fermat's Theorem 99
Sir J. Wilson's 100

SECTION IV.

On the Divisors of Composite Numbers 103
Method of finding a Perfect Number 106

CHAPTER XII.

ON THE INDETERMINATE ANALYSIS.

Solution of the equation $ax \pm by = \pm c$ 107
Its Integer Roots are in Arithmetic Progression 109
Indeterminate Equations of the second degree 110
Formula for the sides of a Right-angled Triangle 112

CHAPTER XIII.

ON CONTINUED FRACTIONS.

To convert a simple fraction into a continued one 113
General Theorem; $p_n q_{n-1} - q_n p_{n-1} = \pm 1$ 115
Application to the solution of an Indeterminate Equation 117
Theory of Intercalation 118
Extraction of the Root of a Number in the form of a Continued Fraction..... 120

CHAPTER XIV.

ON THE RESOLUTION OF COMPOUND FRACTIONS INTO SIMPLE ONES.

General Principle, and different cases 122
Abridgment of the process when some factors of the denominator are known .. 124

CHAPTER XV.

ON LOGARITHMS.

Properties of Logarithms 128
Expansion of a^x..... 129
Series for a Logarithm in terms of its number 131
To change Logarithms from one base to another 133
Advantages of selecting 10 for the base 136
Method of finding the Logarithms of numbers not exactly in the tables, and *vice versa* 137
Use of a Table of Proportional Parts 139
Application of Logarithms to questions on population ..... *ib.*

CHAPTER XVI.

ON THE CALCULATION OF INTEREST AND OF THE VALUE OF ANNUITIES.

Definitions 141
Simple and Compound Interest..... 142
Discount 144
Equation of Payments..... 145
On Annuities 146

CHAPTER XVII.

THE GENERAL THEORY OF SERIES.

SECTION I.

ON RECURRING SERIES.

Developement of a Fraction producing a Recurring Series 150
To find the component Geometric Progressions of a Recurring Series..... 152

Section II.

THE DIFFERENTIAL METHOD OF SERIES.

General Expression for the nth term of a series 155
To find the sum of any series..... ... 156

Section III.

ON THE METHOD OF INCREMENTS.

To find the Increment of a quantity consisting of Factors in Arithmetic Progression 158
To sum a Series by this method 159
Figurate Numbers ..... *ib.*
To find the Increment of the Reciprocal of a Number of Factors in Arithmetic Progression 160
Application to the Summation of Series 161

Section IV.

ON THE LIMITS OF SERIES.

On Geometric Series, considered as limits to more general series..... 162
To find the limits of $(1 + x)^n$ 164

CHAPTER XVIII.

ON THE EXPANSION OF A MULTINOMIAL.

To find the general term of a Multinomial.. 167
Examples 168
To find the number of terms in an expanded Polynomial 169

ADDENDA AND CORRIGENDA.

PAGE

3 After line 2, insert the following explanation;—"$a^{\frac{1}{m}} \cdot a^{\frac{1}{n}}$ is equal to $a^{\frac{m+n}{mn}}$, because it has been shewn, that *each* of these quantities when raised to the $\overline{mn}$th power $= a^{m+n}$; hence the quantities themselves must be equal." A similar explanation will suffice for the equation $(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}$; in line 5.

4 Line 6 from the bottom, for $(-b) + (+d)$ read $(-b) \times (+d)$.

10 Line 12 from the bottom, for $a\sqrt{x} + b\sqrt{y}$ read $a\sqrt{x} \times b\sqrt{y}$.

14 Last line, for "being any number," read "n being any number."

17 Line 10 from the bottom, for $\frac{a}{b^1}$ read $\frac{a}{b}$.

48 In the enunciation of Art. 105, for "*greater*" substitute "*less*," and for "*less*" substitute "*greater*."

95 Line 4 from the bottom, for $\therefore a = \frac{Ab}{B}$, read $\therefore A = \frac{aB}{b}$.

159 Line 7 from the bottom, for "first 5 terms of the 4th order," read "first 5 terms of the 3rd order."

SYLLABUS

OF

ALGEBRA.

DEFINITIONS AND PRINCIPLES.

The term *Algebra*, in its most general sense, is employed to denote the science of general reasoning by means of general symbolical language.

There are two kinds of Algebra, *Arithmetical* and *Symbolical;* and as the latter is founded upon the former, we will first explain the Definitions and Principles of Arithmetical Algebra.

SECTION I.

Definitions and Principles of Arithmetical Algebra.

1. The letters of the alphabet are employed to denote numbers or magnitudes admitting of numerical representation.

2. The signs + and − denote the Arithmetical operations of addition and subtraction: and × and ÷ those of multiplication and division.

3. Multiplication is also denoted by writing quantities with points between them, thus $(a \,.\, b \,.\, c)$, but more frequently the points are omitted.

4. Division is also denoted by writing the dividend over the divisor with a line between them, as $\left(\frac{a}{b}\right)$.

5. The addition of a to itself m times is denoted by ma.

6. The multiplication of a by itself m times is denoted by a^m.

7. Hence when m and n are whole numbers $a^m\, a^n = a^{m+n}$, and $(a^m)^n = a^{mn}$.

For $a^m \cdot a^n = \{a \cdot a \cdot a \ldots\ldots (m \text{ times})\} \times \{a \cdot a \cdot a \ldots\ldots (n \text{ times})\}$

$= a \cdot a \cdot a \ldots\ldots (\overline{m+n} \text{ times})$

$= a^{m+n}$ by the notation assumed in art. 6.

Also, $(a^m)^n = a^m \cdot a^m \cdot a^m \ldots\ldots (n \text{ times})$

$= a \cdot a \cdot a \ldots\ldots (m \text{ times})$
$\times\ a \cdot a \cdot a \ldots\ldots (m \text{ times})$
$\times\ a \cdot a \cdot a \ldots\ldots (m \text{ times})$
$\times \ldots\ldots\ldots\ldots\ldots$
} m itself being repeated n times.

$= a \cdot a \cdot a \ldots\ldots\ldots (mn \text{ times})$

$= a^{mn}$, by assumed notation.

8. One quantity is said to be the square, cube, or nth root, of another, when the first multiplied by itself two, three, or n times, will produce that other quantity, and is thus denoted.

$$\sqrt{a},\quad \sqrt[3]{a},\quad \sqrt[n]{a},\ \&c.$$

9. The n^{th} root of a quantity a is also denoted by $a^{\frac{1}{n}}$.

10. Hence also $a^{\frac{1}{m}} \cdot a^{\frac{1}{n}} = a^{\frac{1}{m}+\frac{1}{n}} = a^{\frac{m+n}{mn}}$ and $(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}$.

For, the first quantity $a^{\frac{1}{m}} \cdot a^{\frac{1}{n}}$ raised to the $\overline{mn}^{\text{th}}$ power

$= a^{\frac{1}{m}} \cdot a^{\frac{1}{m}} \cdot a^{\frac{1}{m}} \ldots\ldots (mn \text{ times}) \times a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \ldots\ldots (mn \text{ times})$

$= a^{\frac{mn}{m}} \cdot a^{\frac{mn}{n}} = a^n \cdot a^m = a^{m+n}$. by Art. 7.

Also the quantity $a^{\frac{m+n}{mn}}$ or $\sqrt[mn]{a^{m+n}}$ raised to the mn^{th} power

$= a^{m+n}$; by Art. 8; hence, $a^{\frac{1}{m}} . a^{\frac{1}{n}} = a^{\frac{m+n}{mn}}$.

Again, $(a^{\frac{1}{m}})^{\frac{1}{n}}$ raised to the n^{th} power $= a^{\frac{1}{m}}$,

and $a^{\frac{1}{mn}}$ raised to the n^{th} power $= a^{\frac{1}{m}}$

$$\therefore (a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}.$$

11. If we have several arithmetical operations to perform upon any quantity, (as for instance to multiply one number by another, then to multiply the result by a second, and afterwards to divide it by a third) it is evidently indifferent in what order we perform these operations.

12. There is no quantity less than zero in Arithmetical Algebra.

13. The addition and subtraction of quantities is performed as in arithmetic, placing the signs + or − between them when they cannot be incorporated together. Art. 2.

14. To subtract $a - b$ from a, we observe, that the result cannot be affected by adding b to each of these quantities, hence

$$a - (a - b) = a + b - (a - b + b)$$
$$= a + b - a.$$

15. The subtraction of $a + b$ from a is impossible, or in other words, there is no such quantity in arithmetical algebra as $- b$. Art. 12.

16. Hence, negative indices have no existence in Arithmetical Algebra.

17. Multiplication,

(1) $a\,(c + d) = a\,c + a\,d$; being the product of $a \times c$, added to the product $a \times d$.

(2) $a\,(c - d) = a\,c - a\,d$; being the difference of the products $a \times c$, and $a \times d$.

(3) $(a + b)(c + d) = (a + b)c + (a + b)d,$

$= ac + bc + ad + bd;$

Here it appears that the result of $(+ b) \times (+ d) = + bd.$

(4) $(a - b)(c - d) = a(c - d) - b(c - d),$

$= ac - ad - (bc - bd),$

$= ac - ad - bc + bd;$ Art. 14.

Here it appears that the result of $(- b) \times (- d) = + bd.$

(5) $(a + b)(c - d) = (a + b)c - (a + b)d,$

$= ac + bc - ad - bd;$

Here we have $(+ b) \times (- d) = - bd.$

(6) $(a - b)(c + d) = (a - b)c + (a - b)d,$

$= ac - bc + ad - bd;$

Here $(- b) + (+ d) = - bd.$

From all which we collect the following,

18. *Rule of Signs;*

The multiplication of two quantities preceded by like signs produces a result preceded by the sign + ; but when they are preceded by unlike signs the result is preceded by the sign — .

SECTION II.

DEFINITIONS AND PRINCIPLES OF SYMBOLICAL ALGEBRA.

19. Symbolical Algebra is a generalization of Arithmetical Algebra, in the same way as this last is a generalization of common Arithmetic. The principles and assumptions of Symbolical, are entirely founded upon those of Arithmetical Algebra, as far as this last science reaches, for this reason the latter is called the *Suggesting Science* to the former.

We shall state briefly the assumptions in Symbolical Algebra, referring in each case to the corresponding one in the Suggesting science.

20. The letters of the Alphabet are assumed to denote any quantities whatever, whether capable of numerical representation or not. Art. 1.

21. The signs + and − are assumed to denote, not only the operations of addition and subtraction, but also any operations whose results are essentially opposite, as for instance, heat and cold, forces pushing and forces pulling, &c. Art. 2.

22. The notation for multiplication and division is the same as in Arithmetical Algebra. Arts. 2, 3, 4, 5 and 6.

23. Hence as in Arithmetical Algebra, when m and n are positive whole numbers $a^m a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$. Art. 7.

24. The equations in the last Article *are assumed* to be true, when m and n are any numbers whatever.

25. The only *assumption* of notation for denoting the n^{th} root of a is $\sqrt[n]{a}$. Arts. 8 and 9.

26. Consistently with the assumptions in Arts. 24 and 25, we may interpret the meaning of fractional and negative indices.

We have $\sqrt[p]{a}$ = such a quantity as multiplied by itself p times will produce a;

Now $a^{\frac{1}{p}}$ multiplied by itself p times

$$= a^{\frac{1}{p}+\frac{1}{p}+\frac{1}{p}\cdots\cdots(p \text{ times})}$$

$$= a^{\frac{p}{p}} = a.$$

Similarly if we had taken a^n instead of a we should have had

$$a^{\frac{n}{p}} = \sqrt[p]{a^n}.$$

Thus we see that the denominator of a fractional index denotes symbolically, the extraction of the root of that order.

Again, the general assumption in Article 24, gives

$$a^{m-n} \cdot a^n = a^{m-n+n} = a^m,$$

and dividing the equals $a^{m-n} \cdot a^n$ and a^m by the quantity a^n, we have,

$$a^{m-n} = \frac{a^m}{a^n}, \text{ or } a^m \cdot a^{-n} = \frac{a^m}{a^n}.$$

Hence we deduce, that to multiply any quantity (as a^m), by a^{-n}, is the same as to divide it by a^n.

(a) If we apply this result to the quantity 1; we shall have,

$$a^{-n} = \frac{1}{a^n}.$$

(b) The above is independant of any particular value of the index, and shews, that any quantity may be transferred from the numerator to the denominator of a fraction, or *vice versa,* by changing the sign of its index.

(c) Hence, $a^0 = a^{m-m} = \frac{a^m}{a^m} = 1$, by the nature of division.

27. In Symbolical Algebra *the order* of any operations to be performed on any quantity is assumed to be indifferent. Art. 12.

28. The minimum of quantity is unlimited; and generally the order of magnitude of quantities is assumed to follow that of the series ..... $-3, -2, -1, 0, 1, 2, 3$ .....

29. When different quantities having the signs + or − prefixed to them, are incorporated by any of the operations of Symbolical Algebra, the sign of the result is determined by the following *assumption:* (Arts. 14 and 18.)

"*When two similar signs come together they are replaced in the result by the single sign + ; but if the signs be dissimilar, by the sign −.*"

30. Since the language and assumptions of Symbolical Algebra are more general than those of Arithmetical Algebra, it frequently happens that symbolical results present themselves in the former science which have no prototypes, or, even meaning, in the latter.

31. *Law of the permanence of equivalent forms.*

The operations of Symbolical Algebra are assumed to be independent of the *specific* values of the symbols employed; hence, if one quantity be shewn to be *equivalent* to another, (for instance $\overline{1+x}|^n$ to $1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 +$ &c. see Art. 24) this equivalency is true whatever be the values of the symbols, so long as they are general in form.

Again, if one quantity be shewn to be equivalent to another in *Arithmetical* Algebra, where the symbols are general in form but specific in value, then, provided there be *any* such corresponding equivalency in *Symbolical* Algebra, it must be the same as the one shewn to be true in the Suggesting science: for, in passing from one to the other no change in *the form* of such equivalencies can take place.

Also, since the laws of the combinations of symbols are *assumed* in Symbolical Algebra, to coincide strictly with the corresponding laws in Arithmetical Algebra; therefore, when an equivalent form is once discovered in the latter, we are at liberty *to assume its existence* also in the former. An instance of this occurs, in *assuming* the equation $a^m \cdot a^n = a^{m+n}$ as *universally* true; (Art. 24.) after having shewn it to be true when m and n are whole positive numbers.

CHAPTER I.

ON THE FUNDAMENTAL OPERATIONS OF ALGEBRA.

Addition.

32. RULE. "Collect the quantities together, prefixing to each its proper sign; and amalgamating the like ones into one term, with a co-efficient equal to the difference (with its proper sign) between the sums of the co-efficients of the positive and negative terms, respectively." Art. 21.

(*a*). This is the Algebraical definition of the *sum* of two quantities, thus $a - b$ is the *Algebraical sum* of a, and, $-b$.

Ex. 2. $a\sqrt{x} + b\sqrt{x} = \sqrt{x}\,(a + b)$, which shews that two surds having the same irrational part $\sqrt{x}$ are added by adding their co-efficients.

SUBTRACTION.

33. RULE. "Change the signs of all the quantities to be subtracted and proceed as in addition."

(*a*). This rule results from the Algebraical definition of the difference between two Algebraical quantities, (see Art. 21.) thus $a + b$ is the result of the *difference* between a and $-b$, since $-(-b) = +b$. Art. 29.

(*b*). When the sign $-$ is placed before several quantities, included between brackets or under a vinculum, it must be under-

stood as affecting the signs of *every* quantity within the brackets, or under the vinculum: the sign of each quantity, when the brackets are removed, being determined by the general principle assumed in article 29.

Thus, $-(a-b-4c+3x) = -a+b+4c-3x$.

MULTIPLICATION.

34. Monomial quantities are multiplied together by multiplying the co-efficients for a new co-efficient, and denoting the multiplication of the quantities themselves by the notation of Articles 2 and 3. Thus $7a \times 6b = 42ab$.

(*a*). The sign of the product must be determined by the principle of signs assumed in Art. 29.

(*b*). The powers of the same quantity are multiplied together by adding the indices. Art. 23.

(*c*). It is indifferent in what order the quantities are multiplied: Art. 27, thus, $b.a = a.b$; they are, however, generally arranged in alphabetical order.

Ex. 1. $a\sqrt{x} + b\sqrt{y} = ab\sqrt{x}\sqrt{y} = ab\sqrt{xy}$,

hence surds having the same exponent are multiplied by multiplying their co-efficients, and also the quantities under the radical sign; retaining the same exponent.

Ex. 2. $ax^{\frac{m}{n}} \times bx^{\frac{p}{q}} = abx^{\frac{m}{n}}x^{\frac{p}{q}} = abx^{\frac{m}{n}+\frac{p}{q}}$.

35. For the multiplication of polynomial quantities, we have the following rule: "Multiply each term of the multiplicand by each term of the multiplier successively, and then collect the partial products as in addition."

Ex. 1. $(a+x)(a-x) = a^2 - x^2$, or the product of the sum and difference of two quantities, is equal to the difference of their squares.

Ex. 2. $(a + x)^2 = a^2 + 2\,a\,x + x^2$,

and $(a + x)^3 = a^3 + 3\,a^2 x + 3\,a\,x^2 + x^3$,

in these cases it may be observed that the result is *symmetrical* with respect to a and x; that is, if x be put for a, and a for x, in the product, the result will remain the same.

36. *To find the square of* $a_1 + a_2 + a_3 + a_4 \ldots\ldots + a_n$.

Let $a_2 + a_3 \ldots\ldots a_n = P$; then the square required

$$= (a_1 + P)^2 = a_1^2 + 2\,a_1 P + P^2.$$

Now the square of $a_1 + a_2 \ldots\ldots a_n$, must be symmetrical with respect to each of the quantities a_1, a_2, a_3, &c. and it appears that it involves a_1^2; hence, it must also involve a_2^2, a_3^2, &c. Secondly, it involves $2\,a_1 P$; that is, $2\,a_1$ multiplied successively into each of the $n - 1$ quantities a_2, a_3, &c.

Again, if we make $P^2 = (a_2 + P')^2$, where

$$P' = a_3 + a_4 + \ldots\ldots a_n,$$

and reason in the same manner with respect to $a_2^2 + 2\,a_2 P'$, we may observe, that we find first a_2^2; and secondly, $2\,a_2$ multiplied into P'; that is, into each of the $n - 2$ quantities a_3, a_4, a_5, &c. and by making $P' = a_3 + P''$, &c. we may go on until we come to $P^{(n)} = a_n$, hence the only quantities involved are the squares of each of the quantities a_1, a_2, &c. and twice the products of every two of them; that is,

$$\begin{aligned}(a_1+a_2+a_3\ldots+a_n)^2 = a_1^2 + \ & a_2^2 + a_3^2 + a_4^2 \ldots\ldots + a_n^2 \\ & + 2\,a_1a_2 + 2\,a_1a_3 + 2\,a_1a_4 \ldots\ldots + 2\,a_1a_n \\ & + 2\,a_2a_3 + 2\,a_2a_4 \ldots\ldots + 2\,a_2a_n \\ & + 2\,a_3a_4 \ldots\ldots + 2\,a_3a_n \\ & + \&c.\end{aligned}$$

a result which, by adding each vertical column, may also be written thus,

$$(a_1 + a_2 + a_3 \ldots\ldots a_n)^2$$

$$= a_1^2 + (2\,a_1 + a_2)\,a_2 + \{2(a_1 + a_2) + a_3\}\,a_3$$

$$+ \{2(a_1+a_2+a_3) + a_4\}\,a_4 \ldots + \{2(a_1+a_2+a_3+\ldots a_{n-1}) + a_n\}\,a_n.$$

(*a*) By similar induction we should find,

$$(a_1 + a_2 + a_3 \ldots + a_n)^3 =$$

$$a_1^3 + (3a_1^2 + 3a_1a_2 + a_2^2)a_2 + \{3(a_1+a_2)^2 + 3(a_1+a_2)a_3 + a_3^2\}a_3 \ldots$$

$$+ \{3(a_1 + a_2 \ldots a_{n-1})^2 + 3(a_1 + a_2 \ldots a_{n-1})a_n + a_n^2\}a_n.$$

(*b*). The above reasoning with respect to symmetrical quantities entering into the product of factors which themselves involve symmetrical quantities, is applicable in many other cases, as a means of abridging the process of actual multiplication. See Peacock's Algebra, p. 27.

37. *To raise* $\sqrt{-1}$ *to any power.*

Every number is either divisible by 4, or must leave a remainder when divided by that quantity, equal to 1, 2, or 3; hence every number is of one of the forms $4m$, $4m+1$, $4m+2$, or $4m+3$; we shall consider each of these cases separately;

(1). $(\sqrt{-1})^{4m}$

$$= \{(\sqrt{-1})^4\}^m = \{(\sqrt{-1})^2 . (\sqrt{-1})^2\}^m = \{(-1).(-1)\}^m$$

$$= 1^m = 1.$$

(2). $(\sqrt{-1})^{4m+1}$

$$= \sqrt{-1}\,(\sqrt{-1})^{4m} = \sqrt{-1} \,.\, 1 \text{ by the first case}; = \sqrt{-1}.$$

(3). $(\sqrt{-1})^{4m+2} = (\sqrt{-1})^2\,(\sqrt{-1})^{4m} = (\sqrt{-1})^2 . 1 = -1.$

(4). $(\sqrt{-1})^{4m+3}$

$$= \sqrt{-1}\,(\sqrt{-1})^{4m+2} = \sqrt{-1}\,(-1) \text{ by the third case}; = -\sqrt{-1}.$$

DIVISION.

38. When the quantities are monomial, we must omit those which are common to the divisor and dividend, and denote symbolically that the remainders are to be divided by each other, by the notation explained in Arts. 3 and 4.

(*a*). Since $(+a) \times (+b) = +ab$; $\therefore \dfrac{+ab}{+a} = +b.$

$$(+a) \times (-b) = -ab; \therefore \frac{-ab}{+a} = -b.$$

$$(-a) \times (-b) = +ab; \therefore \frac{+ab}{-a} = -b.$$

Hence we collect, that if the divisor and dividend be affected with *like* signs, the sign of the quotient is +, but if they be affected with *unlike* signs, the sign of the quotient is −.

(*b*). Any power of a quantity is divided by any other power of the same quantity, by subtracting the index of the divisor from the index of the dividend; for,

$$a^{m-n} = \frac{a^m}{a^n};$$ Art. 26.

(*c*) If the divisor be monomial and the dividend polynomial, each term of the dividend must be divided separately by the divisor.

39. When both the divisor and dividend are polynomial, "arrange each according to the descending powers of some one letter common to both; divide the first term of the divisor by the first term of the dividend, the result is the first term of the quotient; multiply this term into the divisor and subtract the result from the dividend; then consider the remainder (if any) as a new dividend and proceed as before."

40. By this process the product of the divisor and all the terms of the quotient is subtracted from the dividend, and thus the *complete quotient* is obtained when there is no remainder after the last operation, but in some cases the remainder will never disappear and then the quotient is said to be *incomplete*, or in other words, there is no finite expression, which, multiplied into the divisor, will produce the dividend.

Ex. 1. *To divide* $x^n - p\,x^{n-1} + q\,x^{n-2} - r\,x^{n-3} +$ &c. *by* $x - a$.

$$
\begin{array}{l|l}
x^n - p\,x^{n-1} + q\,x^{n-2} - r\,x^{n-3} + \text{\&c.} & x^{n-1} + (a-p)\,x^{n-2} \\
\quad x^n - a\,x^{n-1} & \quad + (a^2 - p\,a + q)\,x^{n-3} + \text{\&c.} \\
\hline
\quad (a-p)\,x^{n-1} + q\,x^{n-2} & \\
\quad (a-p)\,x^{n-1} - (a^2 - p\,a)\,x^{n-2} & \\
\hline
\qquad (a^2 - p\,a + q)\,x^{n-2} - r\,x^{n-3} & \\
\qquad (a^2 - p\,a + q)\,x^{n-2} - (a^3 - p\,a^2 + q\,a)\,x^{n-3} & \\
\hline
\qquad\qquad (a^3 - p\,a^2 + q\,a - r)\,x^{n-3}. & \\
\qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.} &
\end{array}
$$

It is evident that the remainder after the n^{th} operation will be

$$a^n - p\,a^{n-1} + q\,a^{n-2} - r\,a^{n-3} + \&c.$$

which is identical with the dividend, putting a in the place of x; the quotient will be incomplete unless

$$a^n - p\,a^{n-1} + q\,a^{n-2} - r\,a^{n-3} + \&c. = 0.$$

Ex. 2. *To find in what case $x^n - y^n$ is divisible by $x + y$.*

$$\begin{array}{l} x + y)\ x^n - y^n\ (x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \&c. \\ \quad x^n + x^{n-1}y \\ \quad \hline \\ \quad\quad - x^{n-1}y - y^n \\ \quad\quad - x^{n-1}y - x^{n-2}y^2 \\ \quad\quad \hline \\ \quad\quad\quad\quad x^{n-2}y^2 - y^n \\ \quad\quad\quad\quad x^{n-2}y^2 + x^{n-3}y^3 \\ \quad\quad\quad\quad \hline \\ \quad\quad\quad\quad\quad\quad - x^{n-3}y^3 - y^n \\ \quad\quad\quad\quad\quad\quad \&c. \qquad \&c. \end{array}$$

Here it is evident that after the n^{th} operation the remainder (R_n) will be

$$\pm x^{n-n}y^n - y^n,$$

the + or − sign being used as n is even or odd, or

$$R_n = (\pm y)^n - y^n,$$

a quantity which cannot vanish unless the first term be positive, that is, unless n be even, hence in that case only, is $x^n - y^n$ exactly divisible by $x + y$.

In the same way it may be shewn that $\dfrac{x^n - y^n}{x - y}$ is always a complete quotient; that $\dfrac{x^n + y^n}{x + y}$ is so when n is odd; and that $\dfrac{x^n + y^n}{x - y}$ is in no case a complete quotient.

(a). Since

$$\frac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + \ \ldots\ldots + x\,y^{n-2} + y^{n-1}, \text{ exactly,}$$

$\therefore$ by the nature of division,

$$x^n - y^n = (x - y)\,(x^{n-1} + x^{n-2}y \ \ldots\ldots + x\,y^{n-2} + y^{n-1})$$

n being any number.

Similarly, (the upper or lower signs being used as n is odd or even)

$$x^n \pm y^n = (x + y)(x^{n-1} - x^{n-2}y \ldots\ldots \mp x y^{n-2} \pm y^{n-1}).$$

Now if x and y be assumed to represent surds, and n be made of such a value as to render $x^n \pm y^n$ rational, we may in this way find the multiplier which will render any binomial surd $x \pm y$ rational, that multiplier being one of the series,

$$x^{n-1} + x^{n-2}y \ldots\ldots + x y^{n-2} + y^{n-1},$$
$$\text{or } x^{n-1} - x^{n-2}y \ldots\ldots \mp x y^{n-2} \pm y^{n-1}.$$

Ex. To find the quantity which multiplied into the surd $a^{\frac{3}{4}} - b^{\frac{3}{4}}$ will render it rational.

Here $x = a^{\frac{3}{4}}$, $y = b^{\frac{3}{4}}$, and $\therefore n = 4$, and the multiplier is $a^{\frac{9}{4}} + a^{\frac{3}{2}} b^{\frac{3}{4}} + a^{\frac{3}{4}} b^{\frac{3}{2}} + b^{\frac{9}{4}}$, the rationalized result being $a^3 - b^3$.

CHAPTER II.

ON ALGEBRAIC FRACTIONS.

SECTION I.

On the Addition, Subtraction, &c. of Algebraic Fractions.

41. SINCE $\frac{a}{b}$ means the result, of the division of a by b; therefore $\frac{a}{b} \times b = a$: by the nature of division.

42. *The value of a fraction is not altered by multiplying or dividing both its numerator and denominator by the same quantity.*

Let $\frac{a}{b} = x$ be any fraction; then because if equals be multiplied or divided by any quantity, the results are still equal;

$$\therefore \frac{a}{b} \times b = x \times b; \text{ or by (Art. 41) } a = b\,x;$$

hence, $m\,a = m\,b\,x$; or dividing by $m\,b$; $\frac{m\,a}{m\,b} = x = \frac{a}{b}$.

(a). Conversely if we have a fraction $\frac{m\,a}{m\,b}$, the numerator and denominator may be divided by the quantity (m) and the resulting fraction $\left(\frac{a}{b}\right)$, will be equal to the former.

(b). Hence if $\frac{a}{b}$, be a fraction; by multiplying its numerator and denominator by -1, we have $\frac{a}{b} = \frac{-a}{-b}$, which shews that altering the signs of both the numerator and denominator does not alter the value of the fraction.

43. *To reduce fractions to others of equivalent value having a common denominator.*

Multiply each numerator by every denominator, except its own, for the new numerator, and all the denominators for the new common denominator.

For if $\frac{a}{b}$, $\frac{a'}{b'}$, $\frac{a''}{b''}$, &c. be the fractions;

$$\frac{a\,b'\,b''.....}{b\,b'\,b''.....}; \quad \frac{a'\,b\,b''.....}{b\,b'\,b''.....}, \quad \frac{a''\,b\,b'.....}{b\,b'\,b''.....},$$

are other fractions of the same value (by last Art.) but having the denominator $b\,b'\,b''$ .... common to all.

(*a*). If the numerator and denominator of each fraction be multiplied by the least common multiple of all the denominators, the same thing will be effected but the resulting fractions will be in lower terms. See Art. 58.

44. To add or subtract fractions, reduce them to equivalent ones having a common denominator, then add or subtract the numerators, and subjoin the common denominator.

For let $\frac{a}{b}$, $\frac{a'}{b'}$ be any two fractions, and suppose

$$\frac{a}{b} = x, \; \frac{a'}{b'} = x';$$

then $a = b\,x$, and $a' = b'\,x'$, by Art. 41.

Also, $a\,b' = b\,b'\,x$ and $a'\,b = b\,b'\,x'$,

adding or subtracting these equals;

$$a\,b' \pm a'\,b = b\,b'\,x \pm b\,b'\,x' = b\,b'\,(x \pm x');$$

$$\therefore \; x \pm x' = \frac{a\,b' \pm a'\,b}{b\,b'};$$

$$\text{or } \frac{a}{b} \pm \frac{a'}{b'} = \frac{a\,b' \pm a'\,b}{b\,b'}.$$

45. To multiply fractions, multiply the numerators for a new numerator, and the denominators for a new denominator.

For by multiplying together the equals $a = b\,x$ and $a' = b'\,x'$, in the last Article we have

$$a\,a' = b\,b' \times x\,x'\text{; or, dividing by } b\,b',$$

$$x\,x' = \frac{a\,a'}{b\,b'}\text{, that is, } \frac{a}{b} \times \frac{a'}{b'} = \frac{a\,a'}{b\,b'}.$$

(*a*) Hence $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$; and $\left(\frac{a}{b}\right)^{\frac{1}{m}} = \frac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}}$.

46. To divide one fraction by another, invert the numerator and denominator of the divisor, and proceed as in multiplication.

For, by dividing the equals $a\,b' = b\,b'\,x$, and $a'\,b = b\,b'\,x'$, by one another, we have,

$$\frac{a\,b'}{a'\,b} = \frac{b\,b'\,x}{b\,b'\,x'} \text{ (or, by Art. 42, (}a\text{))}, \quad \frac{x}{x'} = \frac{a\,b'}{a'\,b}.$$

That is, $\dfrac{\frac{a}{b}}{\frac{a'}{b'}} = \dfrac{a\,b'}{a'\,b}$.

47. An integer (*c*) may be represented as a fraction by writing it under its equivalent form $\frac{c}{1}$.

48. Hence a fraction is multiplied by an integer,

(1). By multiplying its numerator by the integer: for

$$\frac{a}{b} \times c = \frac{a}{b} \times \frac{c}{1} = \frac{a\,c}{b}\text{, Art. 45.}$$

(2). By dividing the denominator by the integer, for

$$\frac{a}{b} \times c = \frac{c\,a}{b} = \frac{c}{\frac{b}{a}}\text{, by Art. 42 (}a\text{).}$$

49. Also a fraction is divided by an integer,

(1). By dividing its numerator by the integer, for

$$\frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c}\text{(Art. 46)} = \frac{a}{b\,c} = \frac{\frac{a}{c}}{b}\text{, Art. 42 (}a\text{).}$$

(2). By multiplying the denominator by the integer, for

$$\frac{a}{b} \div \frac{c}{1} = \frac{a}{b\,c}. \quad \text{Art. 46.}$$

50. It follows from Articles 45 and 46, that there are four ways by which the fraction $\frac{a}{b}$ may be multiplied or divided by another, $\frac{a'}{b'}$; the operations of each may be thus represented

Multiplication:

(1) $\dfrac{a\,a' \div b'}{b}$; (2) $\dfrac{a\,a'}{b\,b'}$; (3) $\dfrac{a}{b\,b' \div a}$; (4) $\dfrac{a \div b'}{b \div a'}$.

Division:

(1) $\dfrac{a\,b' \div a'}{b}$; (2) $\dfrac{a\,b'}{b\,a'}$; (3) $\dfrac{a}{b\,a' \div b'}$; (4) $\dfrac{a \div a'}{b \div b'}$.

Of these, the second in each case is the most usual, because it is always possible to effect the operations. The others are only used when the divisions indicated can be effected without remainders; when this is the case they are preferable to the second, because they give the result sought in lower terms.

51. The fraction $\frac{1}{2}$ is greater in value than $\frac{1}{4}$; and $\frac{1}{4}$ is greater than $\frac{1}{8}$; it is thus easy to see that if the numerator of a fraction remain the same, while the denominator is increased, the value of the fraction itself is diminished; and if the denominator become infinite the fraction itself becomes $= 0$, or

$$\frac{1}{\infty} = \frac{b}{\infty} = 0; \text{ and conversely, } \frac{1}{0} = \frac{b}{0} = \infty.$$

(*a*). By similar reasoning, $\dfrac{\infty}{b} = \infty$ and $\dfrac{0}{b} = 0$.

(*b*). Hence also $\dfrac{0}{0} = \infty \times 0 = \dfrac{\infty}{\infty} = b$, a quantity which *may* have any finite value.

(*c*). Also, if, $\dfrac{a}{b} = 0$, then a must $= 0$, or $b = \infty$;

if, $\dfrac{a}{b} = \infty$, then a must $= \infty$, or $b = 0$.

SECTION II.

On the Reduction of Algebraic Fractions to their Lowest Terms.

52. A FRACTION is reduced to an equivalent one in its lowest terms, by dividing its numerator and denominator by the *greatest* quantity which measures them both. Art. 42. (*a*).

53. *To find the greatest common measure of two quantities.*

Let A and a be the quantities, arranged according to the powers of some one letter common to both, and of which A is the one containing the highest power of that letter.

Divide A by a, and suppose a contained in A, q times with a remainder ρ; now if ρ be a quantity of the form $R\,r$ where R is a quantity which does not measure both A and a, and therefore forming no part of the common measure required, take the remaining factor r for a new divisor, and divide the last divisor a by it; let the remainder (if any) ρ', be of the form $R'\,r'$; R' being as before, not a measure of both A and a; take r' alone for the new divisor, and divide the last divisor r by it, repeating the process until a divisor (which we will suppose to be r') be found without any remainder, this divisor is the greatest common measure required.

1st. operation a) $A\ (q$

$$\underline{q\,a}$$

$$A - q\,a = \rho = R\,r$$

2nd. operation r) $a\ (q'$

$$\underline{q'\,r}$$

$$a - q'\,r = \rho' = R'\,r'$$

3rd. operation r') $r\ (q''$

$$\underline{q''\,r'}$$

$$r - q''\,r' = 0.$$

To prove this we will premise that if a quantity m, measure another y, it will measure any multiple $t\,y$ of y, by the units in t.

Also, if a quantity m measure two others x and y by the units in t and s so that $t\,m = x$, and $s\,m = y$, then,

$$x \pm y = t\,m \pm s\,m = (t \pm s)\,m;$$

that is, m will measure the sum or difference of those quantities by the units in $t \pm s$.

Now first, r' is a common measure of A and a.

For it measures r by supposition,

and $\therefore$ it measures $q'\,r$ which is a multiple of r,

and ..... $R'\,r'$ which is a multiple of itself.

It therefore measures $R'\,r' + q'\,r$, or a.

It also measures $R\,r$ because it measures r,

and ..... $q\,a$ because it has been proved to measure a.

It therefore measures $R\,r + q\,a$, or A.

And it is obvious that it might in the same way have been proved to measure both a and A, had it been necessary to carry the operation through four or more stages.

Secondly, it is the *greatest* common measure of A and a.

For let x be any other whatever;

then, since x measures A and a, it measures $A - q\,a$, or $R\,r$;

and $\therefore$ it measures r alone, since R is not a divisor of A and a;

and $\therefore$ ..... $a - q'\,r$, or $R'\,r'$;

and also .. .. r' alone.

Hence since *any* other common measure whatever of a and A measures r'; r' must itself be the *greatest* common measure of a and A.

(a). Hence since any other common measure (x) of A and a is a measure of the greatest common measure (r') of A and a; conversely, every quantity which measures r' measures a and A, now the greatest quantity which measures r' is $\frac{1}{1}r'$, the next in

magnitude $\frac{1}{3}r'$, and so on; hence *all* the common measures of a and A are comprised in the series r', $\frac{1}{2}r'$, $\frac{1}{3}r'$, $\frac{1}{4}r'$, &c.

(*b*). Where simple powers only of each letter are involved, this method will not apply, to find the common measure in this case, we must form the co-efficients of each letter successively in the two quantities, until a common divisor is discovered. See Peacock's Algebra, p. 132.

54. In finding the greatest common measure of the fraction

$$\frac{B\,x^m + C\,x^{m-1} + D\,x^{m-2} + \&c.}{b\,x^n + c\,x^{n-1} + d\,x^{n-2} + \&c.}$$

the following modifications must be attended to.

(1). The monomial factors must be first suppressed by the principles explained in Art. 53 (*b*).

(2). The numerator or denominator may be multiplied or divided by any quantity which is not a factor of the other, for by so doing we do not alter the common measure.

(3). By these means B must be made a multiple of b.

(4). In each operation it is unnecessary to carry the divisor so far as to introduce a term into the quotient under a fractional form, for this would be contrary to the supposition made in the exposition of the rule, where the quotients q, q', q'', &c. are all supposed integers.

55. *To find the greatest common measure of three Algebraic quantities, a, b, c.*

Take M the greatest common measure of a and b, so that

$$\alpha M = a, \text{ and } \beta M = b.$$

Take M' the greatest common measure of M and c;

Then M' is the greatest common measure required.

For by hypothesis M' measures c and M;

and $\therefore$ M' measures αM and βM or a and b.

Similarly *any other* measure (x) of M and c may be shewn to measure a and b likewise.

Hence the *greatest* common measure of M and c is likewise the *greatest* of a, b and c.

(a). By similar reasoning:

if x be the greatest common measure of a and b,
y ..... c and d,
z ..... x and y,
then will z ..... a, b, c, and d.

56. *To find the least common multiple of two numbers,* a *and* b.

Let their greatest common measure be x so that $a = f x$, $b = g x$, then $f g x$ is a multiple of both a and b. But since x is the greatest common measure of a and b, that is of $f x$ and $g x$, therefore f and g can have no common divisor; hence $f g x$ is the *least* common multiple of a and b.

But $f = \frac{a}{x}$, $g = \frac{b}{x}$; $\therefore f g x = \frac{a b x}{x^2} = \frac{a b}{x}$,

that is, the least common multiple of two numbers,

$$= \frac{\text{product}}{\text{greatest common measure}}.$$

57. *Every other multiple* (m') *of* a *and* b *is a multiple of* (m) *their least common multiple.*

For if possible let m be contained in m', p times with a remainder r less than m.

Then $m' = p m + r$.

Now a and b measure m' and m, they therefore measure $m' - p m$ or r, that is, a multiple (r), of a and b, has been found less than m, their least common multiple, which is absurd.

(a). Hence all the common multiples of a and b are comprised in the series m, $2 m$, $3 m$, $4 m$, &c.

For the least number which m measures is $2 m$, the next $3 m$, &c.

58. *To find the least common multiple of three numbers,* a, b, *and* c.

Take m the least common multiple of a and b,

m' the least common multiple of m and c.

Then m' is the least common multiple required.

For by hypothesis m' is a multiple of c and m, which last is a multiple of a and b; hence m' is a multiple of a, b, and c.

Again, let x be *any* other multiple of a, b, and c, then because x is a multiple of a and b it is a multiple of m; Art. 57; and because it is a multiple of m and c it is a multiple of m'; Art. 57. hence since *any* multiple of a, b, and c, is a multiple of m', therefore m' must be the *least* common multiple of a, b, and c.

(*a*). By similar reasoning, if m be taken the least common multiple of a and b, m' of c and d, m'' of m and m' then m'' will be the least common multiple of a, b, c, and d.

CHAPTER III.

ON DECIMAL FRACTIONS.

59. If $a, b, c, \ldots p, q, r, s$, be the $n+1$ digits composing any number N, that number may evidently be written thus

$$N = a\,10^n + b\,10^{n-1} + c\,10^{n-2} \ldots + p\,10^3 + q\,10^2 + r\,10 + s.$$

Thus

$$437864 = 400000 + 30000 + 7000 + 800 + 60 + 4$$
$$= 4.10^5 + 3.10^4 + 7.10^3 + 8.10^2 + 6.10 + 4.$$

(*a*). Hence if 10^n be the highest power of 10, which enters into the composition of a number, $n+1$ is the number of its digits.

60. If we divide the number

$$a\,10^n + b\,10^{n-1} \ldots\ldots + q\,10^2 + r\,10 + s,$$

successively by 10, we shall have the following results,

$$a\,10^{n-1} + b\,10^{n-2} \ldots\ldots + q\,10 + r + \frac{s}{10},$$

$$a\,10^{n-2} + b\,10^{n-3} \ldots\ldots + q + \frac{r}{10} + \frac{s}{10^2},$$

&c. &c.

The fractional parts of these numbers, as $\frac{s}{10}$ in the first, $\frac{r}{10} + \frac{s}{10^2}$ in the second, &c. are called *Decimals*, so that generally,

Decimals are fractions having always 10, *or some power of* 10, *for their denominators.*

61. The denominators of decimals are not usually written, but in order to distinguish the numerators of such fractions from whole numbers, they are written with a comma before as many places of figures, as there are cyphers (that is, powers of 10) in the denominator, the deficiency of places (if any) being supplied by cyphers immediately after the comma.

(*a*). Hence, adding cyphers to *the right* of a decimal does not affect its value, for it is only multiplying both numerator and denominator by some power of 10.

62. The value of each figure in a decimal decreases *from* the left, *to* the right, in a tenfold proportion, contrary to the rule which obtains with respect to the digits of integers, for in decimals every succeeding place from the comma introduces another power of 10 into its denominator, (Art. 61) which therefore diminishes the local value of any digit, in that proportion.

63. If a, b, c, d, r, s, &c. be the digits of a decimal D, then

$$D = \frac{a}{10} + \frac{b}{10^2} + \frac{c}{10^3} \ldots\ldots + \frac{r}{10^{n-1}} + \frac{s}{10^n}, \text{ Art. 60.}$$

$$= \frac{a\,10^{n-1} + b\,10^{n-2} + c\,10^{n-3} \ldots\ldots + r\,10 + s}{10^n},$$

the numerator of which is the number having the digits a, b, c, &c.; hence we have the following rule for the conversion of a decimal into its equivalent fraction; "Omit the decimal point and divide the resulting integral number by that power of 10 whose index (n) is the number of decimal places."

64. *To convert a fraction into its equivalent decimal.*

Let $\frac{A}{B}$ be the fraction in its lowest terms.

Then, $\frac{A}{B} = \frac{1}{10^n} \cdot \frac{A\,10^n}{B}$ which is a decimal of n places, the quantity $\frac{A\,10^n}{B}$ must be simplified by actual division; we have then the following rule: "Divide the numerator by the denominator, adding cyphers (that is muliplying by a power of 10) to the numerator, if necessary."

(a). There must be as many (n) decimal places in the result, as there have been cyphers added to the numerator, and to make up this number, if necessary, cyphers must be added to the left of the quotient.

(b). Since $\frac{A \cdot 10^n}{B} = \frac{A\, 2^n\, 5^n}{B}$, unless therefore B, be of the form $2^p\, 5^q$, the division will never terminate, for no power of 2.5 (or 10) will be measured by those prime factors, if 10 itself be not (See Art.); in other words, we can never by adding any number of cyphers, make the result of the division of the numerator by the denominator appear in a finite form.

65. *When a fraction* $\frac{A}{B}$ *is converted into a finite decimal to find the number of decimal places.*

In this case B is of the form $2^p\, 5^q$, Art. 64. (b).

First, let p be $> q$;

$$\therefore \frac{A}{B} = \frac{A}{2^p\, 5^q} = \frac{A\, 5^{-q}}{2^p} = \frac{A\, 5^{p-q}}{2^p\, 5^p} = \frac{A\, 5^{p-q}}{10^p},$$

which is a decimal of p places.

Secondly, let p be $< q$, then

$$\frac{A}{B} = \frac{A}{2^p\, 5^q} = \frac{A\, 2^{-p}}{5^q} = \frac{A\, 2^{q-p}}{2^q\, 5^q} = \frac{A\, 2^{q-p}}{10^q},$$

which is a decimal of q places.

(a). If the fraction $\frac{A}{B}$ be an improper one, the result will consist of a mixed whole number and decimal.

66. *In the conversion of a fraction* $\frac{A}{B}$, *into a decimal, if the division do not terminate, the remainders will recur in a certain order.*

For let B be divided by A, adding cyphers to each remainder according to the rule; Art. 64. Now each remainder must be less than B, consequently there can only be $B - 1$ *different* ones while their number is infinite; Art. 64 (b); hence in the

progress of the division we shall arrive at length at a remainder which is the same as a former one; after this, the circumstances producing the succeeding remainders are the same as those which produced the preceding ones, and therefore the second *period* of remainders will be the same as the first; similarly, the third, fourth, &c. periods will likewise be identical with the first.

Def. Decimals whose digits recur in periods are called, "circulating or repeating decimals."

67. If a fraction of the form $\frac{A}{B' 2^p 5^q}$ be converted into a decimal, the result will be first a finite decimal of p or q places according as p is $>$ or $< q$ (Art. 65), and then a repeating decimal of not more than $B' - 1$ places, (Art. 66).

68. *To reduce a mixed circulating decimal to a vulgar fraction.*

Let N be the non repeating part, containing n digits or places of decimals; P the repeating period containing p places. Then the whole decimal (D) may be thus represented, Art. 63.

$$D = \frac{N}{10^n} + \frac{P}{10^{n+p}} + \frac{P}{10^{n+2p}} + \frac{P}{10^{n+3p}} + \text{&c. } ad\ infinitum.$$

multiply these equals by 10^p;

$$\therefore D\,10^p = N\frac{10^p}{10^n} + \frac{P}{10^n} + \frac{P}{10^{n+p}} + \frac{P}{10^{n+2p}} + \frac{P}{10^{n+3p}} + \text{&c. } ad\ infin.$$

subtract the upper line from the lower,

$$\therefore D(10^p - 1) = \frac{N}{10^n}(10^p - 1) + \frac{P}{10^n}. \quad \text{Divide by } 10^p - 1.$$

$$\therefore D = \frac{N}{10^n} + \frac{P}{10^n(10^p - 1)}.$$

Now $10^p - 1$ is a number consisting of p nines, hence the rule: "A mixed circulating decimal is equivalent to the sum of two fractions, the first of which is the value of the non repeating part considered as a terminating decimal; and the second has for its numerator the repeating period, and for its denominator as many nines as there are digits in the repeating

period, followed by as many cyphers as there are digits in the part not repeating."

(*a*). If the decimal begin to repeat from the *first* digit, $N = 0$; and $n = 0$; and we have in that case,

$$D = \frac{P}{10^p - 1} = \frac{\text{repeating period}}{\text{as many nines as digits}}.$$

69. Decimal fractions are added or subtracted by placing the digits having the same power of 10 for their denominators, under each other, then adding or subtracting as in whole numbers, and placing the decimal point immediately under the other points.

For this is, in effect, only adding or subtracting so many fractions having common denominators; Art. 60.

70. To multiply decimals, multiply them as though they were whole numbers cutting off as many places of decimals as there are places in the multiplier and multiplicand together.

Let D, D' be the decimals having n, n' decimal places respectively.

N, N' the integral numbers arising from omitting the decimal point in each.

$$\text{then } D = \frac{N}{10^n}, \text{ and } D' = \frac{N'}{10^{n'}};$$

$$\therefore DD' = \frac{NN'}{10^{n+n'}};$$

which is a decimal of $n + n'$ places.

71. To divide Decimals, first make the number of decimal places in the dividend at least equal to those in the divisor by adding cyphers to the right of the dividend, when necessary, (Art 61 (*a*).): then divide as in whole numbers, cutting off as many decimal places in the quotient as those in the dividend exceed those in the divisor; supplying the defect (if any) by prefixing cyphers to the left of the quotient.

For, retaining the letters in last Article, we have, when n is $> n'$,

$$\frac{D}{D'} = \frac{N}{10^n} \cdot \frac{10^{n'}}{N'} = \frac{\frac{N}{N'}}{10^{n-n'}}$$

which is a decimal of $n - n'$ places.

If n' be $> n$, and $n' - n = p$, we must according to the rule make the number of places in the dividend equal to those in the divisor, which will be done by adding p places, to the n places which the dividend already possesses; hence,

$$D \text{ or } \frac{N}{10^n} = \frac{N \cdot 10^p}{10^{n+p}};$$

$$\therefore \frac{D}{D'} = \frac{N}{N'} \cdot \frac{10^p \cdot 10^{n'}}{10^{n+p}} = \frac{N}{N'} 10^{n'-n}.$$

which is an integral number.

If $n' = n$, then $\frac{D}{D'} = \frac{N}{N'}$, which is likewise integral; the above rule includes all these cases.

(a). To divide, multiply, &c. circulating decimals they must first be reduced to vulgar fractions by the rule in Art. 68, and then multiplied, &c. by the rules for treating vulgar fractions; though in cases where perfect accuracy is not required, it will be sufficient to take circulating decimals as far as five or six places, and neglecting the remaining places, to consider them as finite and exact decimals.

CHAPTER IV.

ON THE EXTRACTION OF ROOTS.

72. If $N = a\,10^{n-1} + b\,10^{n-2} \ldots\ldots + r\,10 + s$, be a number having n digits, Art. 59, its least possible value will evidently be, when $a = 1$; and all the other digits $= 0$; in that case, the number becomes 10^{n-1}: similarly, 10^n is the *least* number of $n + 1$ digits, &c. It follows from this, that *every* number having n digits will be intermediate in value between 10^{n-1} and 10^n; for this last number being the *least* number having $n + 1$ digits, the next inferior number must be the *greatest* number having only n digits.

73. *The square of every number having* n *digits must have either* 2 n *or* 2 n − 1 *digits.*

For, every number having n digits lies between 10^{n-1} and 10^n by last Article. Therefore the square of every number which has n digits lies between 10^{2n-2} and 10^{2n}.

But 10^{2n-2} is the *least* number having $2n - 1$ digits, by last Article; and 10^{2n} is the *least* number having $2n + 1$ digits; therefore every such square has either $2n - 1$ digits or $2n$.

(*a*). Conversely, if a square number have either $2n$ or $2n - 1$ digits, its root must have n.

For, 10^{2n} is $>$ than the *greatest* number having $2n$ digits; and 10^{2n-2} is the *least* number having $2n - 1$ digits; Art. 72.

Now, the square roots of these last numbers are 10^n and 10^{2n-1}, respectively; the former of which is $>$ the *greatest* number having n digits, and the latter is the *least* number which has n digits.

(*b*). Similarly, if a cube number have $3n$, $3n-1$ or $3n-2$ digits, the cube root must have n.

(*c*). Hence if a number be divided into periods of two, or three, digits each, the number of digits in its square or cube root will be known by the number of periods.

74. *To deduce a rule for the extraction of the square root of a polynomial quantity.*

We have seen, (Art. 36) that,

$$(a_1 + a_2 + a_3 \ldots\ldots a_n)^2 = a_1^2 + \{2a_1 + a_2\} a_2 + \{2(a_1 + a_2) + a_3\} a_3 \ldots\ldots\ldots\ldots\ldots + \{2(a_1 + a_2 + \ldots\ldots a_{n-1}) + a_{n-1}\} a_n.$$

Also the first term of this last quantity is the complete square of a_1; the first two of $a_1 + a_2$, the first three of $a_1 + a_2 + a_3$, &c.

In order therefore to extract the simple quantity $a_1 + a_2 +$ &c. from its square

$$a_1^2 + \{2a_1 + a_2\} a_2 + \text{&c.}$$

it is evident that we must first extract the root a_1 of the first term a_1^2, and subtract its square from the whole quantity; we then observe that the divisor of the second term is $2a_1 + a_2$ or twice the root already found + a new quantity, which new quantity is the second term of the root; by multiplying this divisor by by that second term, and subtracting it, we subtract the complete square of the two first terms; proceeding in this manner we shall, generally, after the $n-1^{\text{th}}$ operation, have subtracted the square of $a_1 + a_2 \ldots\ldots a_{n-1}$ from the quantity proposed, and the n^{th} divisor will be $2(a_1 + a_2 \ldots\ldots a_{n-1}) + a_n$, which is twice the whole root found + a new quantity, that new quantity being the n^{th} term of the root. In this manner may the square root of any Algebraical quantity be found; observing that in all cases the quantity to be operated upon must first be arranged according to the powers of some one letter.

(a). Since, Art. 59. any number may be represented as an algebraical quantity, the rule will be equally applicable to numbers by supposing the quantities a_1, a_2, a_3, &c. the digits of the root, multiplied each by its proper power of 10.

(b). The number whose square root is to be extracted must first be divided into periods by the rule in Art. 73,(c). which will shew how many digits there are in the root, and then the root of the first period must be taken for the first term of the root.

(c). This division into periods must commence from the unit's place, and not from the digit on the left hand, for,
let $A\,10^{2n-2} + B\,10^{2n-3}$ + &c. be a square number of $2n-1$ digits,
$A'\,10^{2n-1} + B'\,10^{2n-2}$ + &c. be a square number of $2n$ digits.

The root of each of these numbers has n digits, Art: 73 (a), and is therefore of the form

$$a\,.\,10^{n-1} + b\,.\,10^{n-2} + \text{\&c.}$$

Now the part $a\,.\,10^{n-1}$ may evidently be found from the part $A\,.\,10^{2n-2}$ in the first case; but in the second, we must take *both* the terms

$$A'\,.\,10^{2n-1} + B'\,.\,10^{2n-2} = 10^{2n-2}(A'\,.\,10 + B),$$

in order that their root may produce the whole of such a term as $a\,.\,10^{n-1}$.

Hence, in determining the root of the first period, we must make the division into periods commence from the *unit's* place.

75. The following scheme will render the application of the rule more intelligible.

$$\begin{array}{rl} & \quad a_1 \quad a_2 \quad a_3 \\ \textit{To extract the root of } 6,10,09 & (\,200 + 40 + 7 \\ a_1^2 = 4\,00\,00 & \text{or } 247 \text{ the root required.} \\ \hline 2\,a_1 + a_2 = 400 + 40) & 2\,10\,09 \\ (2\,a_1 + a_2)\,a_2 = & 1\,76\,00 \\ \hline 2\,(a_1 + a_2) + a_3 = 480 + 7) & 3409 \\ \{2\,(a_1 + a_2) + a_3\}\,a_3 = & 3409 \end{array}$$

In practice the superfluous cyphers are omitted.

76. If $\frac{a}{10^n}$ be a decimal of n places; its square $= \frac{a^2}{10^{2n}}$, and its cube $= \frac{a^3}{10^{3n}}$, which are decimals of $2n$, $3n$, places respectively; the number of places therefore in a decimal whose square or cube root is to be extracted must always be made divisible by 2 or 3, previous to dividing it into periods; this may be done by adding cyphers to the right of the decimal; which, Art. 61 (a), does not alter its value.

77. In the extraction of the square root no period can give a remainder greater than the double of the root obtained; for, if a be the root obtained, $a + 1$ the next integer, then

$$(a + 1)^2 = a^2 + (2a + 1),$$

so that if the remainder exceed by only one unit, twice the root obtained, it will be possible to increase that root by another unit.

78. It will be shewn hereafter (Art. 145, (4)) that if N be not a square number, $\sqrt{N}$ cannot be represented by any finite fraction; the root may however be found to any degree of accuracy, by adding to each remainder periods of two zero's each, and carrying on the result in decimals.

79. We may also find a fraction approaching in accuracy to $\frac{1}{q}th$ of the root of N; and by assuming q sufficiently large, any degree of accuracy may be thus obtained.

Let $\frac{x}{q}$ be such a fraction;

Then $\frac{x}{q}$ must be $\prec N$, and $\frac{x+1}{q} \succ N$;

But $\sqrt{N} = \frac{q}{q}\sqrt{N} = \frac{\sqrt{Nq^2}}{q}$;

Hence, x is $\prec \sqrt{Nq^2}$;

and, $x + 1$ is $\succ \sqrt{Nq^2}$;

Or in other words, x is the greatest integer contained in

$$\sqrt{Nq^2}.$$

80. *When one more than half the number of figures have been extracted, the process may be thus abridged;*

Let N be the number whose root is to be extracted, and having $2n$ or $2n-1$ digits; Art. 73; a the part already known, which is therefore, by supposition, a number having at least n digits; x the part to be determined, and which therefore cannot have more than $\frac{n}{2}-1$ digits;

$$\therefore \sqrt{N} = a + x;$$

$$\text{and } N = a^2 + 2ax + x^2;$$

$$\therefore \frac{N-a^2}{2a} = x + \frac{x^2}{2a};$$

by transposing a^2 and dividing by $2a$.

Now since x can at most have $\frac{n}{2}-1$ digits;

$\therefore x^2$ can at most have $n-2$. Art. 73.

But a is a number having at least n digits;

$\therefore a$ is $> x^2$; and $\therefore \frac{x^2}{a}$ is a proper fraction;

$$\text{consequently, } \frac{x^2}{2a} \text{ is } < \frac{1}{2},$$

and may therefore be neglected if we consider only the integral part of the root;

$$\therefore x = \frac{N-a^2}{2a};$$

$$\text{and the root required} = a + \frac{N-a^2}{2a}.$$

Now $N-a^2$ is the remainder from the operation which has served to determine a. If we denote it by R, the root required will $= a + \frac{R}{2a}$; which is a rule of easy application. For examples see Francœur's complete course, Art. 134.

81. *To deduce a rule for the extraction of the cube root.*

Let $\mathcal{N}$ be the quantity whose root is to be extracted; and suppose that after having divided the number into periods; Art. 73 (*c*), we find a_1 the cube root of the first period.

Let x = the remaining part of the root; then

$$\mathcal{N} = (a_1 + x)^3 = a_1^3 + 3a_1^2 x + 3a_1 x^2 + x^3;$$

$$\therefore \mathcal{N} - a^3 = (3a_1^2 + 3a_1 x + x^2)x; \text{ by transposing } a^3;$$

$$\therefore x = \frac{\mathcal{N} - a_1^3}{3a_1^2 + 3a_1 x + x^2} = \frac{R}{3a_1^2 + 3a_1 x + x^2};$$

denoting the remainder $\mathcal{N} - a_1^3$ by R: now x must be much less than a_1 and therefore $3a_1^2$ must be much greater than $x(3a_1 + x)$; hence we may neglect these terms in comparison with $3a_1^2$ without introducing much error, we shall have then

$$x = \frac{R}{3a_1^2} \text{ nearly,}$$

which quantity we will denote by a_2; having found a_2 we next subtract $(a_1 + a_2)^3$ from $\mathcal{N}$, and calling the remainder R' we find the next part a_3 by taking a number near to

$$\frac{R'}{3(a_1 + a_2)^2} = \frac{R'}{3a_1^2 + (3a_1 + a_2)2a_2 + a_2^2},$$

and proceeding as before, considering $a_1 + a_2$ as one quantity, we at length find the whole root.

Ex. To find the cube root of 986 111 28.

			a_1	a_2	a_3
		98,611,128	(400	+ 60	+ 2
		64000000	$= a_1^3$	[or, 462 the root	
$3a_1^2 =$	480000	3461,1128	$= R$		
$a_2(3a_1 + a_2) =$	75600	:::::::::			
$3a_1^2 + a_2(3a_1 + a_2) =$	555600	33336000	$= a_2\{3a_1^2 + a_2(3a_1 + a_2)\}$		
$a_2^2 =$	3600	::			
$\therefore 3(a_1 + a_2)^2 =$	634800	1275128	$= R'$		
$a_3\{3(a_1 + a_2) + a_3\} =$	2764				
	637564	1275128	$= a_1\{3(a_1 + a_2)^2$		
			$+ a_3(3(a_1 + a_2) + a_3)\}$.		

It appears that the nearest cube root (a_1) of the first period is 400, subtracting the cube of this from the first period, the remainder (R) is 34,000000; to the left of this, $3\,a_1^2$ or 4800 is placed as the imperfect divisor: dividing R by this, the quotient is near to 60, we then in a separate column write $3\,a_1$ or 1200,

$$\begin{aligned} 3\,a_1 &= 1200 \\ a_2 &= \underline{\;\;60} \\ 3\,a_1 + a_2 &= 1260 \\ 2\,a_2 &= \underline{\;120} \\ 3\,(a_1 + a_2) &= 1380 \\ a_3 &= \underline{\;\;\;2} \\ 3\,(a_1 + a_2) + a_3 &= 1382 \end{aligned}$$

add to it a_2 or 60, and multiply this sum 1260 by 60; write the product 75600 under the imperfect divisor, and add these numbers for a perfect divisor, which it appears in this case is 555600. Multiply this by 60 and write the product 33336000 under the first remainder R, and subtract; this, added to $\overline{400}|^3$, which has been already subtracted from the original number, makes up $(a_1 + a_2)^3$ or $(400 + 60)^3$, the remainder 1275128 is therefore what has been denoted by R'. For a new imperfect divisor, we write 60^2 under the last perfect divisor, add this and the two numbers above it, the sum is 3×400^2. Lastly, to the number in the separate column we add 2×60 or 120, the sum 1380 is 3×460. Dividing by the new imperfect divisor we find 2 the next digit of the root, with which we proceed as before. Add 2 to 1380, multiply the sum by 2 and add the product to the imperfect divisor for a new perfect divisor: we then multiply this perfect divisor by 2. When we have subtracted the product we have subtracted 462^3, which as there is no remainder must be the root sought.

(*a*). The rule may be thus stated. Divide the proposed number into periods of three digits each, beginning with the unit's place, Art. 74 (*c*). Find the greatest digit whose cube is contained in the left hand period, this will be the first term (a_1) of the root; subtract a_1^3 from the first period, and to the right of the remainder write the second period to form a dividend; to the left of this write $3\,a_1^2$ to which annex two zero's to form an

imperfect divisor; in a separate column write $3\,a_1$; find a second digit a_2 by dividing the dividend by the imperfect divisor, and annex it to $3\,a_1$ in the separate column; multiply the number in this column as it now stands by a_2, and add the product to the imperfect divisor, this will make the divisor perfect; multiply the perfect divisor by a_2 and subtract the product from the dividend, and bring down the next period. To obtain a new imperfect divisor write a_2^2 under the last perfect divisor, and take the sum of a_2^2 and the two numbers immediately above it; then add $2\,a_2$ to the lowest number in the separate column, and proceed as before until all the digits of the root are obtained.

For examples, see "*Library of Useful Knowledge—Arithmetic and Algebra*, Art. 181, whence the above rule has been taken.

(*b*). If there be a remainder at last, by adding periods of three zero's to it, we may find the decimal parts of the root.

82. The fourth root may be extracted by taking the square root of the square root, for $a^{\frac{1}{4}} = (a^{\frac{1}{2}})^{\frac{1}{2}}$, Art. 24, the fifth root, by taking the square root of the cube root, and so on; but these operations for the higher roots are very seldom employed in practice.

CHAPTER V.

ON EQUATIONS.

83. There are three meanings of the sign = when it is placed between two quantities:

1st. when it expresses that one member is the result of an operation, which on the other side is indicated but not performed as in the example,

$$\frac{a}{1-x} = a + ax + ax^2 + ax^3 + \&c.$$

2nd. When one quantity is Algebraically *equivalent* to the other, as in the example above, where if the series

$$a + ax + ax^2 + \&c.$$

be employed as a factor in multipliction it will produce the same result as when $\frac{a}{1-x}$, is so employed.

3rd. When each member is reducible to *identical expressions;* as in the above example, when the *remainder* is taken into account in the division of a by $1 - x$; in this case, the expressions are *equal* as well as *identical.*

For a further explanation of these principles, see Peacock's Algebra, page 100.

84. In the Algebraical theory of equations, those only are considered, whose members are not reducible to identity with each other. An equation may then be defined to be; "A proposition stating that the result of certain operations performed on certain quantities, is equivalent to the result of certain operations performed on other quantities; the quantities, the

operations, and the equivalency, being expressed by Algebraical symbols."

Def. *A root* of an equation is such a quantity as when substituted for the unknown quantity, *satisfies* the equation, that is, converts it into an *identical* one.

85. The general principle by which the root of an equation may be discovered, is, that, "Any change may be made on the two members of an equation which does not disturb their equality."

86. In accordance with this principle, the following changes are the most frequent:

(1). Any term may be transferred from one side of an equation to the other by changing its sign; for this is in fact only subtracting that term (with its proper sign) from both sides of the equation.

Hence the signs of *all* the terms, on both sides, may be changed.

(2). Both members of an equation may be multiplied or divided by the same or equal quantities.

Hence if an equation contain fractions, and both its members be multiplied by the least common multiple of the denominators of all the fractions, the equation itself will be cleared of fractions.

Also, both members of an equation may be raised to any power; and conversely any root of both may be extracted.

Hence, also an equation may be divided by the whole factor or co-efficient into which the unknown quantity is multiplied; and then the unknown quantity will stand alone.

87. By some of the above processes, any equation may be reduced to the form

$$x^n - p\,x^{n-1} + q\,x^{n-2} - r\,x^{n-3} + \&c. + Qx - R = 0,$$

where the quantities, p, q, &c. do not involve x, but may be positive or negative, whole or fractional.

88. Equations are called simple, quadratic, cubic, biquadratic, &c. according as they involve the 1st, 2nd, 3rd, or 4th, power of the unknown quantity, x.

89. Every simple equation can have but one root. For let $x - R = 0$ be a simple equation; and let α be one root, and if possible, β a different one. Then, putting α for x, see Def. p. 39,

$$\alpha - R = 0; \text{ and also, } \beta - R = 0.$$

$$\therefore \alpha - \beta = 0, \text{ by subtraction; or, } \alpha = \beta,$$

which shews that, in fact, the roots α and β are the same.

90. If $\left.\begin{matrix} a x + b y = c \\ a' x + b' y = c' \end{matrix}\right\}$ be given to find x and y, the methods are three:

(1). Multiply the first equation by an arbitrary quantity k; assume $k a = a'$ or $k = \frac{a'}{a}$, this will render the co-efficients of x equal in both equations, so that x may be eliminated by subtracting one equation from the other.

(2). Obtain from each equation independent values of one unknown quantity in terms of the other, and equate these values.

(3). Find the value of x in terms of y from the first equation, and substitute that value for x in the second.

(a). The result obtained by any one of these methods is

$$x = \frac{c b' - b c'}{a b' - b a'}; \quad y = \frac{a c' - c a'}{a b' - b a'}.$$

91. If there are three equations involving x, y and z, the process is this:

Eliminate z from the two first, this will give an equation between x and y.

Eliminate z from the second and third, this will give another equation between x and y.

Having deduced these two independant equations between x and y, the case is reduced to the one considered in last Article.

QUADRATIC EQUATIONS.

92. The general form of a quadratic equation is

$$x^2 - px = q.$$

By adding another quantity to each side of this equation, we may make the first a complete square; let L be such a quantity;

$$\therefore x^2 - px + L = L + q.$$

Extracting the root on each side, Art. 74, we find the root of the first side is $x - \frac{p}{2}$, with a remainder $L - \frac{p^2}{4}$, which in order to make the first side a complete square, must $= 0$.

Whence, $L = \left(\frac{p}{2}\right)^2$, and $x - \frac{p}{2} = \pm \sqrt{L + q}$.

$$= \pm \sqrt{\left(\frac{p}{2}\right)^2 + q};$$

$$\therefore x = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 + q},$$

which are the roots required.

(*a*). If $p = 0$, the roots are then $\pm \sqrt{q}$, such equations are called *pure quadratics*, the first kind being denominated *adfected.*

(*b*). It appears that since the solution of every quadratic equation involves the extraction of the square root of some quantity, such equations must have *two* roots.

93. It appears from a view of the process in Article 90, that the elimination of one unknown quantity diminishes the number of equations by one. Hence, in order that we may have a final equation to determine the unknown quantity remaining, we must have as many independant equations (that is, equations not deducible from each other) as we have unknown quantities. If there are more equations than unknown quantities, the excess is superfluous; if fewer, the unknown quantities are indeterminate.

94. Quadratic and other higher equations involving more than one unknown quantity may frequently be solved by the principles of elimination given in Arts. 90 and 91. The following is a method more general, though not frequently so convenient as those of Arts. 90 and 91.

Let $F(x, y) = 0$, ... (1).

$f(x, y) = 0$, ... (2).

be two equations involving x and y.

Now there must be as many values of x in these equations as there are of y.

Hence for every *proper* value of y, (that is, values not merely introduced by the process of solution, (see Peacock's Algebra, p. 596.) in these equations, there is a corresponding one of x; if then, *one* of these values of y be substituted in both of the equations (1) and (2), the resulting equations will involve x only; and they will have one, two, or more, values of x common to them both. Let the equations become in consequence of such a substitution

$$F'(x) = 0 \quad \text{and} \quad f'(x) = 0;$$

these equations having *common* values of x will consequently have a common factor involving it.

To find this common factor we must take the two original equations, and divide one by the other, as in the process for discovering the greatest common measure, Art. 53, (excluding throughout fractional quotients and remainders) until we obtain a remainder involving y only; and by equating this remainder to zero, we shall obtain a value of y, which when substituted in the last divisor (involving x) will make that divisor a common factor of equations (1) and (2). And by finding *all* such values of y we may thus obtain all the *proper* values of x.

(*a*). The above process will not apply

1st. When the original equations are incompatible with each other, this is indicated by the last remainder or the last divisor, becoming a numerical quantity which cannot be made $=0$.

2nd. When the original equations have a common factor; this is indicated by the final equation, involving y only, becoming an identical one: we must in that case exclude the common factor from the two first equations and recommence the process.

3rd. Two, three, or more values of x may correspond to the same value of y in the two first equations, in which case the common factor of $F'(x)$ and $f'(x)$ will be a quadratic, cubic, or higher equation in x; we must then make the last, two last, &c. divisors $=0$, in order to obtain the corresponding *proper* value of y. See *Francœur's complete course*, Art. 523.

(*b*). The process of finding the highest common divisor of $F'(x)$ and $f'(x)$ may introduce factors into the resulting equation in y, (and therefore values of y in that equation) which are foreign to the original equations, but this can never happen when the original equations do not exceed the second degree. See *Peacock's Algebra*, *p.* 646.

For examples in the preceding theory, and a more full explanation of the principles upon which it is founded, see *Peacock's Algebra*, p. 641 *et seq.* Also, *Francœur's complete Course*, Art. 521, where several examples are given.

95. Different artifices may frequently be resorted to, which will often be much shorter than the general process of elimination, explained in the last Article.

(1). When the sum of the dimensions of the unknown quantities in every term of each equation is the same, make $y = t\,x$

and substitute this value for y, this will in general simplify the equations.

(2). When the unknown quantities are similarly involved, assume $x = z + v$, and $y = z - v$; by these substitutions we shall obtain simpler equations to solve in z and v, than the original ones in x and y: having found z and v, x and y will be known from them.

(3). Divide one equation by the other, when this can be done without leaving remainders.

CHAPTER VI.

ON INEQUALITIES.

96. An inequality is a proposition which expresses Algebraically that one quantity is greater ($>$), or, less than ($<$), another.

97. If $A > B$ then $B < A$; so that if the members of an inequality be transposed, the sign of inequality must be reversed.

98. Let A be $> B$ and $= B + \delta$;

$$\therefore A + L = B + L + \delta;$$

L being any arbitrary quantity;

$$\therefore A + L > B + L.$$

Similarly $A - L > B - L$.

Hence we see that any quantity may be added to, or subtracted from both members of an inequality, and the same species of inequality will still subsist.

99. *A fortiori*, each member of the inequality may be added to itself any number of times, or if $A > B$, then $pA > pB$, and conversely, $\frac{1}{p} A > \frac{1}{p} B$.

(a). Again, if $pA > pB$ then A is $> B$, so that an inequality may be divided by any factor common to both its members.

(b). Also if $A > B$ then $A^m > B^m$ and $A^{\frac{1}{m}} > B^{\frac{1}{m}}$.

100. A quantity may be transferred from one member of an inequality to the other, by changing its sign, for this is the same as subtracting it from both sides. Art. 98.

101. If $A > B$; then by transposition, $-B > -A$; and $\therefore -A < -B$.

Hence it appears that if the signs of *both* members be changed the inequality must be reversed.

(*a*). Conversely, if the inequality be multiplied or divided by a negative quantity (which would change the signs of both members) the inequality is reversed. That is, if $A > B$, then $-pA < -pB$, and similarly, $-\frac{1}{p}A < -\frac{1}{p}B$.

102. It is evident that if $A > B$ and $A' > B'$, then
$A + A' > B + B'$, and $A - A' > B - B'$.

(*a*). Also, that if $A - \delta > B$ or $A > B + \delta$, then $A > B$.

CHAPTER VII.

ON RATIOS, PROPORTION, AND VARIATION.

SECTION I. *On Ratios.*

103. THE term *ratio* is used to express the relation which exists between two quantities of the same kind, with respect to magnitude.

(*a*). Hence if a and b be two magnitudes, the fraction $\frac{a}{b}$ will represent their ratio; which is also denoted thus; $(a : b)$.

104. A ratio is said to be a ratio *of greater or less inequality*, according as the *antecedent*, (or first term); is greater or less, than the *consequent;* (or last term).

105. *A ratio of greater inequality is increased, and of less inequality diminished, by adding any quantity to both its terms.*

Let $\frac{a}{b}$ be the first, and $\frac{a+x}{b+x}$ the second ratio, found by adding the quantity x to each term; the first ratio will be greater or less than the second, according as, $\frac{a}{b}$ is $>$ or $<\frac{a+x}{b+x}$;

that is as

$$\frac{a\,b+a\,x}{b\,(b+x)} \text{ is } > \text{ or } < \frac{a\,b+b\,x}{b\,(b+x)}, \text{ Art. 44.}$$

$$\text{or as } a\,x \text{ is } > \text{ or } < b\,x, \text{ Art. 98.}$$

$$\text{or as } a \text{ is } > \text{ or } < b, \text{ Art. 99. } (a).$$

that is, according as the primitive ratio, $\frac{a}{b}$, is one of greater or less inequality. Art. 104.

106. If $\frac{a}{b}$ and $\frac{c}{d}$ be two ratios, then, $\frac{a\,c}{b\,d}$ is called *the sum* of these ratios or the ratio *compounded* of them: also $\frac{a^2}{b^2}$ is called *the duplicate* ratio of $\frac{a}{b}$.

107. If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \frac{e}{f} =$ &c.

Then the compound ratio is $\frac{a\,b\,c\,d\,e}{b\,c\,d\,e\,f}$ which in this case $= \frac{a}{f}$. Art. 42 (a).

108. If the terms of a ratio be large quantities, differing but little from each other, we may find their double, treble, &c. ratio, in an easier manner than by actual involution.

Let $\frac{a \pm \delta}{a}$ be the ratio, where the difference δ is very small compared with a, then n times this ratio

$$= \frac{(a \pm \delta)^n}{a^n} \text{ (Art. 106.)} = \left(\frac{a \pm \delta}{a}\right)^n = \left(1 \pm \frac{\delta}{a}\right)^n$$

$$= 1 \pm n\frac{\delta}{a} + \frac{n\,(n-1)}{1\,.\,2}\left(\frac{\delta}{a}\right)^2 \pm \text{&c.}$$

by the binomial theorem. See Art. 138.

Now the fraction $\frac{\delta}{a}$ is very small by hypothesis; therefore not much error will be introduced by neglecting its square and other higher powers; the compound ratio will then

$$= 1 \pm n\frac{\delta}{a} \text{ nearly} = \frac{a \pm n\,\delta}{a}.$$

SECTION II.

ON PROPORTION.

109. *Proportion* is the equality of ratios; thus if the ratio $\frac{a}{b}$ be equal to the ratio $\frac{c}{d}$, then the four quantities a, b, c, d, are said to be *proportionals;* this is also expressed by writing the quantities thus; $a : b :: c : d$; or, $a : b = c : d$.

110. Granting that $a : b :: c : d$; or, $\frac{a}{b} = \frac{c}{d}$; the following propositions are true:

(α). *The product of the extremes = that of the means.*

For if $\frac{a}{b} = \frac{c}{d}$ $\therefore a\,d = b\,c$.

Conversely if $a\,d = b\,c$, then $a : b :: c : d$.

(β). *Invertendo.* $b : a :: d : c$.

For divide unity by each of the equal quantities $\frac{a}{b}$, and $\frac{c}{d}$,

$$\therefore \frac{1}{\left(\frac{a}{b}\right)} = \frac{1}{\left(\frac{c}{d}\right)} \text{ or } \frac{b}{a} = \frac{d}{c}; \text{ that is, } b : a :: d : c.$$

(γ). *Alternando.* $a : c :: b : d$.

For $a\,d = b\,c$ by (α); divide these equals by $d\,c$,

$$\therefore \frac{a}{c} = \frac{b}{d}, \text{ or } a : c :: b : d.$$

(δ). *Componendo.* $a + b : b :: c + d : d$.

For, adding 1 to each side of the equation $\frac{a}{b} = \frac{c}{d}$,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1, \text{ or } \frac{a+b}{b} = \frac{c+d}{d};$$

that is, $a + b : b :: c + d : d$.

(ε). *Dividendo.* $a - b : b :: c - d : d.$

For, $\frac{a}{b} - 1 = \frac{c}{d} - 1$, or $\frac{a-b}{b} = \frac{c-d}{d}$;

that is, $a - b : b :: c - d : d.$

(ζ). *Convertendo.* $a : a - b :: c : c - d.$

For, *invertendo* (by β) $\frac{b}{a} = \frac{d}{c}$; $\therefore 1 - \frac{b}{a} = 1 - \frac{d}{c}$,

$$\text{or } \frac{a-b}{a} = \frac{c-d}{c};$$

and, invertendo again, $\frac{a}{a-b} = \frac{c}{c-d}$,

$$\text{that is, } a : a - b :: c : c - d.$$

(η). Since by (δ) $\frac{a+b}{b} = \frac{c+d}{d}$, and by ($\varepsilon$) $\frac{a-b}{b} = \frac{c-d}{d}$,

by dividing the first result by the second, we have,

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}; \text{ or } a + b : a - b :: c + d : c - d.$$

(θ). $ma : mb :: nc : nd$; *where* m *and* n *are any quantities whatever.*

$$\text{For, } \frac{a}{b} = \frac{c}{d}; \therefore \frac{ma}{mb} = \frac{nc}{nd}, \text{ Art. 42.}$$

that is, $ma : mb :: nc : nd.$

Hence, conversely, any factor common either to the first and second, or to the third and fourth terms of a proportion may be expunged.

(ι). $ma : nb :: mc : nd$;

For since, $\frac{a}{b} = \frac{c}{d}$; by multiplying both sides by $\frac{m}{n}$, we have, $\frac{ma}{nb} = \frac{mc}{nd}$. That is, $ma : nb :: mc : nd.$

Hence also, conversely, any factor common, either to the first and third, or to the second and fourth, terms of a proportion may be expunged.

(κ). $ma \pm nb : pa \pm qb :: mc \pm nd : pc \pm qd.$

For $\frac{ma}{nb} = \frac{mc}{nd}$; by ($\iota$);

hence, componendo or dividendo,

$$\frac{ma \pm nb}{nb} = \frac{mc \pm nd}{nd}.$$

Similarly, $\frac{pa \pm qb}{qb} = \frac{pc \pm qd}{qd}$;

and dividing one of these results by the other,

$$\frac{ma \pm nb}{pa \pm qb} \cdot \frac{qb}{nb} = \frac{mc \pm nd}{pc \pm qd} \cdot \frac{qd}{nd},$$

$$\text{or, } \frac{ma \pm nb}{pa \pm qb} = \frac{mc \pm nd}{pc \pm qd}.$$

By giving to m, n, p, and q, different values, this result may be made to include all the previous propositions on proportion.

(λ). $a^{\frac{m}{n}} : b^{\frac{m}{n}} :: c^{\frac{m}{n}} : d^{\frac{m}{n}}.$

For if, $\frac{a}{b} = \frac{c}{d}$; $\therefore \left(\frac{a}{b}\right)^{\frac{m}{n}} = \left(\frac{c}{d}\right)^{\frac{m}{n}}$, Art. 86. (2).

(μ). If besides the proportion $\frac{a}{b} = \frac{c}{d}$ we have several other proportions as

$$\frac{a'}{b'} = \frac{c'}{d'}; \quad \frac{a''}{b''} = \frac{c''}{d''}, \text{ \&c.}$$

then, multiplying these equations together, we have

$$\frac{a\,a'\,a''}{b\,b'\,b''} = \frac{c\,c'\,c''}{d\,d'\,d''},$$

$$\text{or} \qquad a\,a'\,a'' : b\,b'\,b'' :: c\,c'\,c'' : d\,d'\,d''.$$

This is called *compounding* the proportions; and common terms appearing in the result may be expunged by (θ) or (ι).

(a). If $a : b :: c : d :: e : f :: g : h ::$ &c. then,

$$a : b :: a + c + e + g + \text{\&c.} :: b + d + f + h + \text{\&c.}$$

For, $a\,b = b\,a$,
$a\,d = b\,c$,
$a\,f = b\,e$,
$a\,h = b\,g$,
&c. = &c.
} by Art. 110 (a).

$\therefore\ a\,(b + d + f + h + \&c.) = b\,(a + c + e + g + \&c.)$,

or, Art. 110 (a).

$a : b :: a + c + e + g + \&c. : b + d + f + h + \&c.$

SECTION III.

ON VARIATION.

111. Def. (1). If there be two quantities x and y, which constantly change in value, x is said to vary as y (denoted thus, $x \propto y$) when the new values of each respectively, are in continued proportion. Thus if x', and y' be the new values of x and y, then $x \propto y$, if, $x : y :: x' : y'$.

(2). If $x \propto \frac{1}{y}$, x is said to vary *inversely* as y; and conversely, y is said to vary inversely as x.

(3). If $x \propto yz$; x is said to vary *jointly* as y and z.

112. *If* $x \propto y$, *and* $y \propto z$; *then* $x \propto z$.

For $x : x' :: y : y'$
and $y : y' :: z : z'$ } by hypothesis.

$\therefore$ $x : x' :: z : z'$ Art. 110 (μ).

That is, $x \propto z$.

113. *If* $x \propto z$ *and* $y \propto z$; *then also* $\sqrt{xy} \propto z$ *and* $x \pm y \propto z$.

For $x : x' :: z : z'$
$y : y' :: z : z'$

$\therefore$ $xy : x'y' :: z^2 : z'^2$
or $\sqrt{xy} : \sqrt{x'y'} :: z : z'$, Art. 110 ($\lambda$).
that is, $\sqrt{xy} \propto z$.

Again, since, $x : x' :: z : z' :: y : y'$;
$\therefore$ alternando, $x : y :: x' : y'$;
componendo or dividendo, $x \pm y : y :: x' \pm y' : y'$;
alternando, $x \pm y : x' \pm y' :: y : y'$;
$:: z : z'$.

That is, $x \pm y \propto z$.

114. *If* x $\propto$ y *and* z *be any other quantity variable or invariable, then* z x $\propto$ z y.

For, since, $x : x' :: y : y'$;

and, $z : z' :: z : z'$;

$\therefore$ Art. 110 (*c*). $xz : x'z' :: yz : y'z'$;

that is, $xz \propto yz$.

(*a*). If z be constant, we have similarly, $x \propto zy$, and if it be assumed so that $x' = zy'$, we shall then have, $x = zy$; so that, in all cases, if one quantity vary as another, it is equal to that other, multiplied by some constant quantity.

(*b*). Hence, a variation may be operated upon as an equation, without destroying the truth of the variation.

(*c*). If $x \propto y$, dividing each by y, $\frac{x}{y} \propto 1$, or is constant.

(*d*). Also, if $x \propto yz$, then dividing by y, $z \propto \frac{x}{y}$.

115. If $x \propto y$ and $X \propto Y$ then $Xx \propto Yy$; for

$$x : x' :: y : y'$$
$$X : X' :: Y : Y'$$
$$\therefore Xx : X'x' :: Yy : Y'y';$$
that is, $Xx \propto Yy$.

116. If a variable quantity is to be changed by the operation of two or more causes it is indifferent whether we consider all those causes as acting together, or whether we consider a partial change to be effected by one cause, then another partial change to be effected by another, and so on, considering the effect of all the causes separately. Suppose the variation of x to depend upon the variations of y_1, y_2, y_3, y_4, &c. in the following manner,

$$\left.\begin{array}{l} x \propto y_1 \\ x \propto y_2 \\ x \propto y_3 \\ x \propto y_4 \end{array}\right\} \text{when} \left\{\begin{array}{l} y_2,\ y_3, \text{ and } y_4 \\ y_1,\ y_3, \text{ and } y_4 \\ y_1,\ y_2, \text{ and } y_4 \\ y_1,\ y_2, \text{ and } y_3 \end{array}\right\} \text{are constant,}$$

Then when all these quantities are variable

$$x \text{ will } \propto y_1 \cdot y_2 \cdot y_3 \cdot y_4.$$

For, let us consider the changes separately; and let

$$x \text{ become } x_1 \text{ when } y_1 \text{ becomes } y'_1;$$

then, $x_1 \ldots\ldots x_2 \ldots\ldots y_2 \ldots\ldots y'_2;$

$x_2 \ldots\ldots x_3 \ldots\ldots y_3 \ldots\ldots y'_3;$

$x_3 \ldots\ldots x_4 \ldots\ldots y_4 \ldots\ldots y'_4;$

then x_4 will be the final change which x undergoes in consequence of the variations of *all* the quantities y_1, y_2, y_3, and y_4; and, by compounding all these proportions, and striking out common factors, Art. 110 (μ), we shall have,

$$x : x_4 :: y_1 \cdot y_2 \cdot y_3 \cdot y_4 : y'_1 \cdot y'_2 \cdot y'_3 \cdot y'_4;$$

$$\text{or } x \propto y_1 \cdot y_2 \cdot y_3 \cdot y_4.$$

(a). It is upon this principle that questions in both the single and double rule of three are solved; x being put for the *effect* produced, and y_1, y_2, &c. for the *agents* or causes, which produce that effect. See Peacock's Algebra, p. 324.

CHAPTER VIII.

ON PROGRESSIONS.

117. QUANTITIES are said to be in *Arithmetic Progression*, when they increase or decrease by a common difference: and in *Geometric Progression* when they increase or decrease by a common multiplier, *or ratio.*

118. *To find the sum of a series of quantities in Arithmetic Progression.*

Let $a =$ first term, b the common difference, n the number of terms, s the aggregate sum, then by the definition,

$$s = a + (a + b) + (a + 2b) \ldots . + (a + \overline{n-1}\, b),$$

and writing the terms in a reverse order,

$$s = (a + \overline{n-1}\, b) + (a + \overline{n-2}\, b) + (a + \overline{n-3}\, b) \ldots . + (a + b) + a;$$

now add the terms of these two lines together as they stand,

$$\therefore\ 2s = (2a + \overline{n-1}\, b) + (2a + \overline{n-1}\, b) + \text{&c. to } n \text{ terms.}$$

$$= n(2a + \overline{n-1}\, b);$$

$$\therefore\ s = \frac{n}{2}\{2a + (n-1)b\} = \frac{n}{2}(a + y),$$

putting $y = a + (n-1)b =$ last term.

(*a*). Hence the rule for summing an Arithmetic Progression is: "add the first and last terms together, and multiply by half the number of terms."

(*b*). Any three of the quantities s, a, n, or b, being given, the fourth may be found from the equation,

$$s = \{2a + (n-1)b\}\frac{n}{2}.$$

(*c*). Also, if a, y, and s be given, b may be found by eliminating n from the equations,

$$\left.\begin{aligned} s &= \{2a + (n-1)b\}\frac{n}{2} \\ y &= a + (n-1)b \end{aligned}\right\} \quad \text{whence, } b = \frac{y^2 - a^2}{2s - (a+y)}.$$

119. *To insert* p *arithmetic means between* u *and* v.

Here, $n = p + 2$; also, $v = u + (n-1)b = u + (p+1)b$;

$$\therefore b = \frac{v-u}{p+1};$$

hence the means are found by adding once, twice, &c. this value of b to u, we have then,

$$\text{1st mean} = u + \frac{v-u}{p+1} = \frac{pu+v}{p+1}.$$

$$\text{2nd mean} = u + 2\frac{v-u}{p+1} = \frac{(p-1)u+2v}{p+1}.$$

$$\text{3rd mean} = u + 3\frac{v-u}{p+1} = \frac{(p-2)u+3v}{p+1}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$p\text{th mean} = u + p\frac{v-u}{p+1} = \frac{u+pv}{p+1}.$$

(*a*). If $p = 1$ the mean is $\frac{u+v}{2}$.

120. *To find the sum of a series of quantities in Geometric Progression.*

Let a = first term, S = sum, r = common ratio, y = last term; then,

$$S = a + ar + ar^2 \ldots\ldots + ar^{n-2} + ar^{n-1};$$

$$\therefore Sr = ar + ar^2 + ar^3 \ldots\ldots ar^{n-1} + ar^n,$$

and by subtracting the upper line from the lower

$$Sr - S = -a + ar^n;$$

$$\therefore S = a\frac{r^n - 1}{r-1} = \frac{ry - a}{r-1}.$$

(*a*). Any three of the quantities r, y, a, S, being given the fourth may be found.

(*b*). Also n may be found by means of logarithms, when a, r, and S are given.

121. If the number of terms be indefinitely continued; r being a proper fraction, each term will be less than the preceding one, and in that case, since by going far enough we may find a term less than any magnitude which can be assigned, there will be some quantity to which the sum of the terms approaches but never actually reaches, let this be σ.

Then $\sigma = a + ar + ar^2 + ar^3 + \ldots\ldots$ *in infinitum;*

$\therefore\ \sigma r = ar + ar^2 + ar^3 + ar^4 \ldots\ldots +$ *in infinitum.*

hence $\sigma - \sigma r = a$

$\therefore\ \sigma = \frac{a}{1-r}$

(*a*). If we divide a by $1 - r$ we shall re-produce the indefinite quotient

$$a + ar + ar^2 + \&c.$$

hence the expression $\frac{a}{1-r}$ is *symbolically equivalent* to this quotient. See Art. 83.

122. *To insert* p *Geometric means between* u *and* v.

Here, number of terms $= p + 2$.

Also, $v = u\,r^{n-1}$;

$$\therefore\ r = \left(\frac{v}{u}\right)^{\frac{1}{n-1}} = \left(\frac{v}{u}\right)^{\frac{1}{p+1}},$$

and the means are

$$u\left(\frac{v}{u}\right)^{\frac{1}{p+1}},\ u\left(\frac{v}{u}\right)^{\frac{2}{p+1}},\ \&c.$$

the p^{th} being, $u\left(\frac{v}{u}\right)^{\frac{p}{p+1}}$.

(*a*). If $p = 1$, the mean is $u\sqrt{\frac{v}{u}} = \sqrt{uv}$.

123. The principle employed in summing a Geometric series (Art. 120.) may be employed in finding the sums of many other series which follow a similar law.

Ex. To sum a series of fractions whose numerators are in Arithmetic and denominators in Geometric Progression.

$$\text{Let } S = a + \frac{a+b}{r} + \frac{a+2b}{r^2} \ldots\ldots + \frac{a+(n-1)b}{r^{n-1}};$$

$$\therefore \frac{S}{r} = \frac{a}{r} + \frac{a+b}{r^2} + \frac{a+2b}{r^3} \ldots\ldots + \frac{a+(n-1)b}{r^n},$$

$$\text{and } S - \frac{S}{r} = a + \frac{b}{r} + \frac{b}{r^2} \ldots\ldots + \frac{b}{r^{n-1}} - \frac{a+(n-1)b}{r^n}$$

$$= a + \frac{b}{r^{n-1}} \cdot \frac{r^{n-1}-1}{r-1} - \frac{a+(n-1)b}{r^n}. \quad \text{Art. 120.}$$

$$\therefore S = \frac{ra}{r-1} - \frac{a+(n-1)b}{r^{n-1}(r-1)} + \frac{b(r^{n-1}-1)}{r^{n-2}(r-1)^2}.$$

124. Three quantities are said to be in Harmonic progression, when the first is to the third as the difference between the first and second, is to the difference between the second and third.

125. *The reciprocals of quantities in Harmonic Progression are in Arithmetic Progression.*

For if a_1, a_2, a_3, a_4, &c. be in harmonic progression, then

$$a_1 : a_3 :: a_1 - a_2 : a_2 - a_3;$$

$$\therefore a_3 a_1 - a_3 a_2 = a_1 a_2 - a_1 a_3, \quad \text{Art. 110. (a).}$$

and, dividing by $a_1 a_2 a_3$,

$$\frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \delta, \text{ suppose,}$$

Similarly, if $a_2 : a_4 :: a_2 - a_3 : a_3 - a_4$, then,

$$\delta = \frac{1}{a_3} - \frac{1}{a_2} = \frac{1}{a_4} - \frac{1}{a_3};$$

hence the quantities $\frac{1}{a_1}$, $\frac{1}{a_2}$, $\frac{1}{a_3}$, $\frac{1}{a_4}$, &c. have a common difference (δ), and are therefore in Arithmetic Progression.

(a). Hence to insert p Harmonic means between u and v, we must insert p Arithmetic means, between $\frac{1}{u}$ and $\frac{1}{v}$.

(b). If $p = 1$ the mean is (Art. 119. (a).) $\frac{\frac{1}{u} + \frac{1}{v}}{2} = \frac{u + v}{2\,u\,v}$.

(c). Hence the Arithmetic, Geometric, and Harmonic means between two quantities, u and v, form a Geometric Progression, whose common ratio is

$$\frac{2\sqrt{u\,v}}{u + v}.$$

126. Let a_1, a_2, a_3, be in Arithmetic progression, then, by Art. 117.

$$a_1 - a_2 = a_2 - a_3, \text{ and } \therefore 1, \text{ or } \frac{a_1}{a_1} = \frac{a_1 - a_2}{a_2 - a_3}.$$

Also, if a_1, a_2, a_3 be in Geometric progression, having the common ratio r, then

$$a_3 = r\,a_2 = r^2 a_1, \ \therefore \frac{a_1}{a_2} = r = \frac{a_2}{a_3}.$$

Hence, by Art. 109. (ζ). $\frac{a_1}{a_2} = \frac{a_1 - a_2}{a_2 - a_3}$.

The three progressions may therefore be thus defined,

$$\frac{a_1 - a_2}{a_2 - a_3} \begin{cases} = \frac{a_1}{a_1}, \text{ in the Arithmetic.} \\ = \frac{a_1}{a_2}, \text{ in the Geometric.} \\ = \frac{a_1}{a_3}, \text{ in the Harmonic.} \end{cases}$$

CHAPTER IX.

ON PERMUTATIONS AND COMBINATIONS.

SECTION I.

The quantities considered being all dissimilar.

127. Def. 1. THE different *orders* in which *the whole* of any number of quantities can be arranged are called their *permutations.*

Def. 2. The different orders in which any number, less than the whole, can be arranged, are called their *variations.*

Def. 3. The different *collections* that can be formed of any number of quantities, without regard to the *order* in which they are placed, are called their *combinations.*

Def. 4. The number of *permutations* of m things is denoted thus; $({}^{m}P)$; the number of *variations* (taking them p and p together) thus, $({}^{m}V_{p})$; and their *combinations* (taking them p and p together) thus, $({}^{m}C_{p})$.

128. *To find the number of variations of* m *things taken* p *and* p *together;* $({}^{m}V_{p})$.

In m things $a_1, a_2, a_3, \ldots . a_m$; a_1 may be placed before each of the rest, and thus form $m - 1$ variations, (taken two and two) of which a_1 stands first: similarly, a_2, by being placed before each of the rest, will form $m - 1$ variations; and so of the rest; so that there are on the whole $m(m - 1)$ variations taken two and two together; or, ${}^{m}V_{2} = m(m - 1)$.

Again, since it appears that in m things there are $m\,(m-1)$ variations taken two and two together; for m write $\overline{m-1}$; hence, out of $m-1$ things, (taken two and two) there are $(m-1)\,(m-2)$ variations; and by prefixing a_1 to each of these there will be $(m-1)\,(m-2)$ variations, taken three and three, of which a_1 stands first; the same may be said of a_2, a_3, &c. so that there are, on the whole, $m\,(m-1)\,(m-2)$ variations of m things taken three and three, or, ${}^{m}V_3 = m\,(m-1)\,(m-2)$.

By following the same reasoning it appears that

$${}^{m}V_p = m\,.\,(m-1)\,(m-2)\;\ldots\ldots\;(m-\overline{p-1}).$$

(a). If $p = m$, or the variations become permutations, we have,

$${}^{m}P = m\,(m-1)\;\ldots\ldots\;3\,.\,2\,.\,1 = 1\,.\,2\,.\,3\;\ldots\ldots\;m.$$

(b). The above proceeds upon the supposition that each quantity can only enter *once* into each variation; consequently, such a variation as, $a_1\,a_1$; or as, $a_1\,a_2\,a_2$, &c. would be rejected.

129. *To find the number of combinations of* m *things taken* p *and* p *together.* $({}^{m}C_p)$.

Now, ${}^{m}V_p = m\,(m-1)\,(m-2)\;\ldots\ldots\;(m-\overline{p+1})$, by last Article.

But *each* of the combinations in ${}^{m}C_p$ admits of as many variations as the p quantities in *one* of those combinations will form, that is, of $1\,.\,2\,.\,3\;\ldots\ldots\;p$; Art. 128. ($a$)., hence,

$${}^{m}C_p \times 1\,.\,2\,.\,3\;\ldots\ldots\;p = {}^{m}V_p$$

$$= m\,(m-1)\,(m-2)\;\ldots\ldots\;(m-\overline{p-1}),\ \text{Art. 128.}$$

$$\therefore\ {}^{m}C_p = \frac{m\,(m-1)\,(m-2)\;\ldots\;(m-\overline{p-1})}{1\,.\,2\,.\,3\;\ldots\ldots\;p}.$$

(a). Hence,

$${}^{m}C_2 = m\,\frac{m-1}{2};\ \text{and,}\ {}^{m}C_3 = \frac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3}.$$

(*b*). Hence also, the sum of the number of combinations of m things, taken 1 and 1, 2 and 2, 3 and 3, ... m and m, or,

$$^mC_1 + {}^mC_2 + {}^mC_3 \ldots\ldots + {}^mC_m$$
$$= m + \frac{m\,(m-1)}{1\,.\,2} + \frac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3} + \&c.$$
$$= (1+1)^m - 1 = 2^m - 1,$$

as will be shewn hereafter, (Art. 139. e.)

(*c*). If we have m quantities a_1, a_2, a_3, ... a_m, and we form a single combination by taking p of them, the remaining $m - p$ quantities will form another and *supplementary* combination, and this will be the case for every combination formed by taking p quantities out of m; hence, there will be as many such *supplementary* combinations as there are primary ones, or in other words, $^mC_p = {}^mC_{m-p}$.

This might also have been proved by putting $m - p$ for p in the expression for ^mC_p in Art. 129.

(*d*). Hence, if the number of combinations of m things taken 1 and 1, 2 and 2, 3 and 3, &c. be written successively, the same values will recur in reverse order beyond the middle term.

130. Since, (Art. 129),

$$^mC_p = \frac{m\,(m-1)\,\ldots\,(m-\overline{p-2})\,(m-\overline{p-1})}{1\,.\,2\,.\,3\,\ldots\ldots\,(p-1)\,.\,p}.$$

$\therefore$ writing $p-1$ for p,

$$^mC_{p-1} = \frac{m\,(m-1)\,\ldots\ldots\,(m-\overline{p-2})}{1\,.\,2\,.\,3\,\ldots\ldots\,(p-2)\,(p-1)};$$

$$\therefore\ ^mC_p = {}^mC_{p-1} \times \frac{m-p+1}{p}.$$

Hence, in forming these combinations, we may deduce each from the next preceding, by multiplying it by the fraction

$$\frac{m-p+1}{p}.$$

See *Francœur's complete course*, Art. 477.

$$(a).\quad {}^{m}C_{p}+{}^{m}C_{p-1}=\frac{m\ldots.(m-\overline{p-2})(m-\overline{p-1})}{1\,.\,2\ldots.(p-1)\,.\,p}+\frac{m\ldots(m-\overline{p-2})}{1.2\ldots.(p-1)}$$

$$=\frac{m\ldots.(m-\overline{p-2})}{1\,.\,2\ldots..(p-1)}\left\{\frac{m-p+1}{p}+1\right\}.$$

$$=\frac{(m+1)\,m\ldots.(m-\overline{p-2})}{1\,.\,2\,.\,3\ldots.p}$$

$$={}^{m+1}C_{p},$$

a result which is useful in proving the binomial theorem ; Art. 138.

131. Each succeeding combination will be greater than the preceding one, up to a certain value of p, when they will begin to recur in a reverse order ; Art. 129 (d). ; if p be the greatest, it must be greater than either the preceding or succeeding one ; or,

${}^{m}C_{p}$ must be $>$ ${}^{m}C_{p-1}$ and also $>$ ${}^{m}C_{p+1}$; that is,

$$\frac{m\ldots..(m-\overline{p-2})\;(m-\overline{p-1})}{1\,.\,2\ldots...(p-1)\,p}\text{ is }>\frac{m\ldots.(m-\overline{p-2})}{1\,.\,2\ldots...p-1},$$

$$\text{and also }>\frac{m\ldots...(m-\overline{p-1})\,(m-p)}{1\,.\,2\ldots...p\,(p+1)};$$

hence, striking out common factors, Art. 99 (a).

$$\frac{m-p+1}{p}>1\text{ and also }>\frac{(m-p+1)\,(m-p)}{p\,(p+1)}.$$

Hence, from the first inequality, we have, $p<\frac{m+1}{2}$,

and from the second inequality, we have, $p>\frac{m-1}{2}$,

1st. When m is even, these conditions are fulfilled by making $p=\frac{m}{2}$.

2nd. When m is odd, $\frac{m+1}{2}$ and $\frac{m-1}{2}$ are consecutive numbers, and since we cannot have a whole number p, intermediate in value between two consecutive numbers, we must in

that case have p equal to them both, so that then there will be *two* maximum combinations equal to each other; these may easily be shewn to be greater than the preceding and succeeding combinations, from the general expression for ${}^{m}C_{p}$, Art. 129.

132. *Supposing a class of* r *things*, $a_1, a_2, a_3, \ldots\ldots a_r$, *be given; and another class of* s *things*, $b_1, b_2, b_3, \ldots\ldots b_s$; *to find the number of combinations that can be formed by always taking* ρ *from the first class and* σ *from the second.*

For the first class we have ${}^{r}C_{\rho} = \frac{r(r-1)\ldots\ldots(r-\overline{\rho-1})}{1.2.3\ldots\ldots\rho}$;

for the second class, we have ${}^{s}C_{\sigma} = \frac{s(s-1)\ldots\ldots(s-\overline{\sigma-1})}{1.2.3\ldots\ldots\sigma}$.

Now *each* of the combinations ${}^{r}C_{\rho}$ is to be coupled with each of ${}^{s}C_{\sigma}$.

Hence the number required $= {}^{r}C_{\rho} \times {}^{s}C_{\sigma}$

$$= \frac{r(r-1)\ldots\ldots(r-\overline{\rho-1})}{1.2.3\ldots\ldots\rho} \times \frac{s(s-1)\ldots\ldots(s-\overline{\sigma-1})}{1.2.3\ldots\ldots\sigma}.$$

It is evident that each combination consists of $\rho + \sigma$ quantities.

(a). To find the number of these combinations containing a_1 without a_2.

Here we have $r = 2$, $\rho = 1$, and the number

$$= {}^{r}C_{\rho} \times {}^{s}C_{\sigma} = {}^{2}C_{1} \times {}^{s}C_{\sigma} = 2 \,.\, {}^{s}C_{\sigma}.$$

If $p = \rho + \sigma =$ number of quantities entering into one of these combinations; and $m = r + s =$ whole number of quantities in the two sets a_1, a_2, &c. b_1, b_2, &c. together; then $2 \,.\, {}^{s}C_{\sigma}$ becomes ${}^{m-2}C_{p-1} \times 2$.

(b). To find the number of combinations of m things taken p and p, which always contain two of the three specified ones, a_1, a_2, and a_3.

Here the number s of one set is $m - 3$; the number r, of the other is 3; out of which ρ (or 2) are always to be selected; hence $p = \rho + \sigma = 2 + \sigma$, and $\therefore \sigma = p - 2$; the number required is, therefore,

$$^{r}C_{\rho} \times {}^{s}C_{\sigma} = {}^{3}C_{2} \times {}^{m-3}C_{p-2} = 3 \times {}^{m-3}C_{p-2}.$$

For more examples see *Francœur, Art.* 478.

133. *To find the number of variations on the same hypothesis as that in the last Article.*

Here, each combination in $^{r}C_{\rho} \times {}^{s}C_{\sigma}$ will admit of

$1 . 2 . 3 \ldots\ldots p$ variations, where $p = \rho + \sigma$,

hence the number required

$$= 1 . 2 . 3 \ldots\ldots p \times {}^{r}C_{\rho} \times {}^{s}C_{\sigma}.$$

(*a*). If the quantities a_1, a_2, a_3, &c. in the first class are to occupy only particular and previously assigned places, in each variation, then each combination in $^{r}C_{\rho} \times {}^{s}C_{\sigma}$ will admit only of the variations in respect of the s quantities, b_1, b_2, b_3, &c. of the other class, not so restricted; hence, on this hypothesis, the number of variations will

$$= 1 . 2 . 3 \ldots\ldots \sigma \times {}^{s}C_{\sigma} \times {}^{r}C_{\rho} = {}^{s}V_{\sigma} \times {}^{r}C_{\rho}. \quad \text{Art. 129.}$$

(*b*). If the ρ quantities, taken each time out of the class of a's, be restricted from being changed amongst themselves in the places assigned to them, so that they are not to occupy any of those places indifferently, the preceding expression must be multiplied by $1 . 2 . 3 \ldots\ldots \rho$.

It then becomes $= {}^{s}V_{\sigma} \times {}^{r}V_{\rho}$. Art. 129.

(*c*). The theory in this and the last Article, may easily be extended to the cases where three or more classes of things are given, to select from.

CHAP. X. SECTION II.

ON PERMUTATIONS AND COMBINATIONS, WHEN THE QUANTITIES CONSIDERED ARE NOT ALL DISSIMILAR.

134. *To find the number of variations of* m *things, taken* n *and* n *together, on the supposition that each quantity may enter* 1, 2, 3, ... *or* n *times into each variation.*

First, the quantity a_1 may be placed before *itself*, or before any one of the other (m) quantities, and thus form m variations. The same may be said of $a_2, a_3, \ldots\ldots a_m$, so that there will be on the whole $m \,.\, m$ or m^2 variations, taken two and two.

Again, a_1 may be placed before each of the foregoing variations, and thus form m^2 variations, taken three and three, in which a_1 stands first; the same may be said of each of the remaining quantities $a_2, a_3, \ldots\ldots a_m$, so that there will be on the whole $m \,.\, m^2$ or m^3 variations, taken three and three.

Similarly, there will be m^n, when taken n and n.

(a). The whole number of variations of m things, on this supposition, taken 1 and 1, 2 and 2, 3 and 3 ..... n and n will

$$= m + m^2 + m^3 \ldots\ldots + m^n = m\,\frac{m^n - 1}{m - 1}. \quad \text{Art. 120.}$$

135. *To find the number of permutations of* m *things (taken* m *and* m *together)* r *of which are alike.*

If we have a set of r identical quantities (as a set of a's) then evidently ${}^rP = 1$, there being only one possible way of arranging them. If now we introduce *a new quantity* a_1, this may occupy the first, second, third, &c. or, $\overline{r+1}^{\text{th}}$ (that is last)

place in the above set, and thus form $r + 1$ different permutations; if we introduce a second *different* quantity a_2, this may occupy the first, second, third, &c. or $\overline{r+2}^{\text{th}}$ (that is, last) place, in *each* of the above permutations, and thus form on the whole $(r+1)(r+2)$ permutations.

Similarly, the introduction of *a third quantity* a_3, would cause $(r+1)(r+2)(r+3)$ permutations; and of $m - r$ *different* quantities,

$$(r+1)(r+2)\ldots\ldots(r+\overline{m-r-1})(r+\overline{m-r})$$

permutations; that is, (r of the m quantities being alike);

$$^{m}P = (r+1)(r+2)\ldots\ldots(r+\overline{m-r-1})(r+\overline{m-r})$$

$$= \frac{1.2.3\ldots\ldots r\,(r+1)(r+2)\ldots\ldots(m-1)\,m}{1.2.3\ldots\ldots r}$$

(since r is supposed less than m)

$$= \frac{1.2.3\ldots\ldots(m-1)\,m}{1.2.3\ldots\ldots r}.$$

(*a*). By similar reasoning, if, in addition to r of the quantities being alike, we have s, t, &c. of them alike also, then,

$$^{m}P = \frac{1.2.3.4\ldots\ldots m}{1.2.3\ldots\ldots r \times 1.2.3\ldots\ldots s \times 1.2.3\ldots\ldots t \times \&c.}$$

136. *To find the number of combinations of* m *things* $a_1, a_2, a_3, \ldots\ldots a_m$, *taken* p *and* p, *on the supposition that each quantity may enter* 1, 2, 3, ... *or* p *times into each combination.*

This would be taking into account such combinations as $a_2 a_2 a_3 a_3$, $\sqsubset a_4 a_1 a_1 a_2$, &c. and the number will evidently be the the same as the number of terms in the expanded polynomial $(a_1 + a_2 + a_3 \ldots\ldots + a_m)^p$, the like terms being amalgamated together by addition.*

* It will easily be seen, by multiplying the quantity

$$a_1 + a_2 + a_3 \ldots\ldots + a_m,$$

two, or three times by itself, and attentively considering the result, that no one combination of the quantities a_1, a_2, is omitted, which answers the conditions in the text. See also Francœur's complete Course, Art. 494.

Hence, (as will be shewn in the Chapter on Series) the number required is the m^{th} term of the $\overline{p+1}^{\text{th}}$ order of figurate numbers, and its value

$$= (p+1)\frac{p+2}{2} \ldots\ldots \frac{p+n-1}{n-1}.$$

Here, m may be $>$, $=$, or $<$, p.

(a). To obtain the sum of the combinations, taken 1 and 1, 2 and 2, &c. we must sum a series of figurate numbers.

137. *To find the number of combinations of* m *things, taken* 1 *and* 1, 2 *and* 2, ... m *and* m *together, when there are* α *alike of one class,* β *of another,* γ *of another, &c.*

Suppose there are a, a's; β, b's; γ, c's, &c. then the problem will evidently be the same as to find all the divisors of such a quantity as $a^{\alpha}\, b^{\beta}\, c^{\gamma}$ ..... where $\alpha + \beta + \gamma$ ..... $= m$.

Now, 1, a, a^2, a^3 ..... a^{α} are all divisors.

Also, 1, b, b^2, b^3, ... b^{β} .....

and, 1, c, c^2, c^3, ... c^{γ} .....

Likewise, every possible combination of these quantities; hence all the divisors will be comprised in the terms of the expression

$(1+a+a^2 \ldots\ldots + a^{\alpha})(1+b+b^2 \ldots\ldots + b^{\beta})(1+c+c^2 \ldots\ldots + c^{\gamma}) \ldots\ldots$

and their number will

$$= (1+\alpha)\,(1+\beta)\,(1+\gamma) \ldots\ldots$$

(a). To find the number of those which contain b, they will evidently be those comprised in the expression

$(1+a+a^2 \ldots\ldots + a^{\alpha})\,(b+b^2+b^3 \ldots\ldots + b^{\beta})(1+c+c^2 \ldots\ldots c^{\gamma}) \ldots\ldots$

and their number will, therefore,

$$= (1+\alpha)\,\beta\,(1+\gamma) \ldots\ldots$$

CHAPTER X.

ON THE BINOMIAL THEOREM.

138. If we multiply together the three factors

$$(x + a_1)(x + a_2)(x + a_3)$$

the result is,

$$x^3 + (a_1 + a_2 + a_3)x^2 + (a_1 a_2 + a_1 a_3 + a_2 a_3)\, x + a_1 a_2 a_3$$
$$= x^3 + c_1 x^2 + c_2 x + c_3;$$

by denoting the combinations (not their *number* as in Art. 129) of n things (which in this case $= 3$), taken 1 and 1, 2 and 2, &c. by c_1, c_2, &c.

Let us assume then, generally, that,

$$(x + a_1)(x + a_2) \ldots\ldots (x + a_n)$$
$$= x^n + c_1 x^{n-1} + c_2 x^{n-2} \ldots + c_p x^{n-p} \ldots + c_{n-1} x + c_n \ldots\ldots (\alpha)$$

and multiply this equation by $x + b$;

$$\therefore (x + a_1) \ldots\ldots (x + a_n)(x + b)$$
$$= x^{n+1} + (c_1 + b)x^n + (c_2 + bc_1)x^{n-1} + \ldots\ldots + (c_p + b\, c_{p-1})\, x^{n-p+1} \ldots + c_n b,$$

which (see Art. 130 (α).

$$= x^{n+1} + c'_1\, x^n + c'_2\, x^{n-1} \ldots\ldots c'_p + x^{n-p+1} \ldots\ldots + c'_{n-1} x + c'_n,$$

(denoting the combinations of $n + 1$ things, taken 1 and 1, 2 and 2, p and p, &c. by c'_1, c'_2, c'_p, &c.).

Hence, if the assumption (α) be a correct one when there are n factors, it appears that it is also correct when there are $n + 1$; but it is shewn by actual multiplication to be correct when there are two, or three, factors, it is therefore so when there are four, five, &c. and therefore generally correct.

Now, let $a_1 = a_2 = a_3 \ldots\ldots = a_n = a$, in equation (a);

then, $c_1 = n\,a$, $\sqsupset c_2 = \dfrac{n\,(n-1)}{1\,.\,2}\,a^2$, $\sqsupset c_3 = \dfrac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,a^3, \ldots$

$$c_p = \frac{n\,(n-1)\ldots\ldots(n-\overline{p-1})}{1\,.\,2\,.\,3\,\ldots\ldots\,p}\,a^p \text{ Art. 129, hence}$$

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1\,.\,2}x^{n-2}a^2 \ldots + \frac{n(n-1)\ldots(n-\overline{p-1})}{1\,.\,2\,.\,3\,\ldots\,p}x^{n-p}a^p$$

$$\ldots\ldots + n\,a^{n-1}x + a^n$$

If for a we write $-a$, we shall have,

$$(x-a)^n$$

$$= x^n - nax^{n-1} + \ldots + (-a)^p \cdot \frac{n(n-1)\ldots(n-\overline{p-1})}{1\,.\,2\,.\,3\,\ldots\,p}x^{n-p} \ldots + (-a)^n.$$

This is the binomial theorem when n is an integer.

139. In proving the theorem in the cases in which n is fractional or negative, we will first observe that

$$(x' + a)^n = a^n\left(1 + \frac{x'}{a}\right)^n = a^n\,(1+x)^n,$$

by putting x for $\dfrac{x'}{a}$; hence we may confine the proof to the more simple form for $(1+x)^n$.*

---

* The following is a general proof of the binomial theorem without employing the principle of the permanence of equivalent forms; it is not, however, so strict, or so satisfactory as the one in the text.

Let

$(1+x)^n = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.$ square this equation.

$(1+x)^n = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.$

$$\therefore (1+x)^{2n} = \left.\begin{array}{l} 1 + a_1 x + a_2 x^2 + a_3 x^3 + \&c. \\ \quad + a_1 x + a_1^2 x^2 + a_1 a_2 x^3 + \&c. \\ \quad\quad + a_2 x^2 + a_1 a_2 x^3 + \&c. \\ \quad\quad\quad + a_3 x^3 + \&c. \\ \quad\quad\quad\quad + \&c. \end{array}\right\} \quad (1).$$

Now, first, if there be any *general* form of the series for

But

$$(1+x)^{2n} = (1+\overline{2x+x^2})^n$$

$$= 1 + a_1(2x+x^2) + a_2(2x+x^2)^2 + a_3(2x+x^2)^3 + \&c.$$

and equating co-efficients of similar powers of x in these two series for $(1+x)^{2n}$, we have,

$$2a_1 = 2a_1, \quad a_1^2 + 2a_2 = a_1 + 4a_2, \quad \therefore a_2 = \frac{a_1^2 - a_1}{2} = \frac{a_1(a_1-1)}{1.2};$$

$$2(a_3 + a_1 a_2) = 4(a_2 + 2a_3),$$

$$\therefore a_3 = \frac{1}{3}a_2(a_1-2) = \frac{a_1(a_1-1)(a_1-2)}{1.2.3}; \quad \&c. = \&c.$$

hence

$$(1+x)^n = 1 + a_1 x + \frac{a_1(a_1-1)}{1.2}x^2 + \frac{a_1(a_1-1)(a_1-2)}{1.2.3}x^3 + \&c. \ldots (2)$$

Now, since, $(1+x)^n = 1 + a_1 x + \&c.$ and, from (1).

$$(1+x)^{2n} = 2 + 2a_1 x + \&c.$$

it appears that if n be made $= 2n$, then a_1 becomes $= 2a_1$.

Hence, also, it appears that the value of a_1 is dependant on that of n. Let us suppose that

$$a_1 = \alpha + \beta n + \gamma n^2 + \&c.$$

multiply this by 2;

$$\therefore 2a_1 = 2\alpha + 2\beta n + 2\gamma n^2 + \&c.$$

but $2a_1 = \alpha + 2\beta n + 4\gamma n^2 + \&c.$

by making $n = 2n$, and $a_1 = 2a_1$;

hence, comparing similar powers of n in the two last series,

$$2\alpha = \alpha, \quad \therefore \alpha = 0, \quad 2\gamma = 4\gamma, \quad \therefore \gamma = 0, \quad \&c.$$

$$\therefore a_1 = \beta n \text{ and } (1+x)^n = 1 + \beta n x + \&c.$$

Let

$$n = 1, \quad \therefore 1 + x = 1 = \beta x, \quad \therefore \beta = 1,$$

and finally

$$a_1 = \beta n = n;$$

hence, replacing a_1 by n in (2),

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^2 + \&c.$$

n being positive or negative, integer or fractional.

$(1 + x)^n$, it must be the one proved to hold in the case of whole numbers, otherwise that form would not be general.

Secondly, by *assuming* that there is such a general form, we do not, from that assumption, derive any consequences inconsistent with the meaning already assigned, (Art. 26.) to negative and fractional indices; for since

$$(1 + x)^n (1 + x)^{n'} = (1+x)^{n+n'};$$ Art. 23.

hence, expanding each of these quantities, supposing n and n' positive integers, we shall have, by the former part of the proof; Art. 138;

$$\left(1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \&c.\right)\left(1 + n'x + \frac{n'(n'-1)}{1 \cdot 2} x^2 + \&c.\right)$$

$$= 1 + (n + n')x + \frac{(n + n')(n + n' - 1)}{1 \cdot 2} x^2 + \&c. \ldots\ldots (\beta).$$

Now, the series composing the right hand side of this equation, being *the form* of the product of the two composing the left, when n and n' are whole numbers, will also be *the form* of the product of these series, when n and n' are any numbers whatever, for the rules for the multiplication, &c. of symbolical quantities, do not depend upon any *specific* values of those quantities. See Art. 31.

1st. In equation (β) let $n' = -n$, then we have from that equation,

$$\left(1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \&c.\right)\left(1 + n'x + \frac{n'(n'-1)}{1 \cdot 2} x^2 + \&c.\right) = 1.$$

But since n is positive and integral, we have, from Art. 138,

$$1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \&c. = (1 + x)^n,$$

hence

$$(1 + x)^n \left(1 + n'x + \frac{n'(n'-1)}{1 \cdot 2} x^2 + \&c.\right) = 1,$$

and, dividing by $(1 + x)^n$,

$$1 + n'x + \frac{n'(n'-1)}{1 \cdot 2} x^2 + \&c. = \frac{1}{(1 + x)^n} = (1 + x)^{-n},$$

for n' put its value $-n$; and we shall have,

$$(1+x)^{-n} = 1 + (-n)\,x + \frac{(-n)\{(-n)-1\}}{1\,.\,2}\,x^2 + \&c.$$

$$= 1 - n\,x + \frac{n\,(n+1)}{1\,.\,2} - \&c.;$$

this shews that the theorem is not inconsistent when a negative value is given to the index n.

2ndly. The reasoning by which we deduced equation (β), when n and n' were any quantities whatever, will equally apply to shew, that the following equation is true; n', n'', n''', &c. being each positive or negative fractions,

$$(1+n\,x+\&c.)(1+n'x+\&c.)\,(1+n''x+\&c.)\,(1+n'''x+\&c.)\,(\&c.)$$

$$= 1 + r\,x + \frac{r\,(r-1)}{1\,.\,2}\,x^2 + \&c. \text{ where } r = n + n' + n'' + n''' + \&c.$$

Now, suppose, $n = n' = n'' = n''' = \&c$; and let q be the number of repetitions; so that, $r = q\,n$; then,

$$(1 + n\,x + \&c.)^q = 1 + q\,n\,x + \frac{q\,n\,(q\,n-1)}{1\,.\,2}\,x^2 + \&c.;$$

and since n is supposed fractional, let $n = \frac{p}{q}$; p being a positive or negative *integer*.

$$\therefore\ (1 + n\,x + \&c.)^q = 1 + p\,x + \frac{p\,(p-1)}{1\,.\,2}\,x^2 + \&c.$$

which, by the first case, since p is supposed integer $= (1+x)^p$; and extracting the q^{th} root on each side,

$$1 + n\,x + \frac{n\,(n-1)}{1\,.\,2}\,x^2 + \&c. = (1+x)^{\frac{p}{q}}.$$

and, replacing n by its value $\frac{p}{q}$, we have

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}\,x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{1\,.\,2}\,x^2 + \&c.$$

This proves the theorem when the index is fractional.

(*a*). When n is a positive integer the series will end after the $\overline{n+1}^{\text{th}}$ term, since then, the $\overline{n+2}^{\text{th}}$ and all succeeding co-efficients will involve the factor

$$\frac{n\,(n-1)\,(n-2)\,\ldots\ldots\,(n-\overline{n-1})\,(n-n)}{1\,.\,2\,.\,3\,\ldots\ldots\,n\,(n+1)\,(n+2)} = 0;$$

but if n be fractional or negative the series will never end,

(*b*). It appears from Art. 131 that if n be even, the greatest co-efficient will be the middle one, this, being the $\left(\frac{n}{2}+1\right)^{\text{th}}$, will

$$= \frac{n\,(n-1)\,(n-2)\,\ldots\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\,.\,3\,\ldots\ldots\,\frac{n}{2}}$$

$$= \frac{1\,.\,2\,.\,3\,\ldots\ldots\,\frac{n}{2} \times n\,(n-1)\,(n-2)\,\ldots\ldots\left(\frac{n}{2}+1\right)}{\left(1\,.\,2\,.\,3\,\ldots\ldots\,\frac{n}{2}\right)^2}$$

$$= \frac{1\,.\,2\,.\,3\,\ldots\ldots\,n}{\left(1\,.\,2\,.\,3\,\ldots\ldots\,\frac{n}{2}\right)^2} = \frac{(1\,.\,3\,.\,5\,\ldots\ldots\,\overline{n-1})\,(2\,.\,4\,.\,6\,\ldots\ldots\,n)}{\left(1\,.\,2\,.\,3\,\ldots\ldots\,\frac{n}{2}\right)^2}$$

$$= \frac{(1\,.\,3\,.\,5\,\ldots\ldots\,\overline{n-1})\,.\,2^{\frac{n}{2}}\,.\left(1\,.\,2\,.\,3\,\ldots\ldots\,\frac{n}{2}\right)}{\left(1\,.\,2\,.\,3\,\ldots\ldots\,\frac{n}{2}\right)^2}$$

$$= \frac{1\,.\,3\,.\,5\,\ldots\ldots\,(n-1)}{1\,.\,2\,.\,3\,\ldots\ldots\,\frac{n}{2}}\,2^{\frac{n}{2}}.$$

(*c*). When n is odd, there will be two maximum co-efficients equal to each other, Art. 131, these being the

$$\overline{\frac{n+1}{2}}^{\text{th}} \text{ and } \overline{\frac{n+3}{2}}^{\text{th}},$$

will be found by putting these quantities for p in the general

expression for the p^{th} co-efficient; their values may, as in (b), be reduced to the form

$$\frac{1 \cdot 3 \cdot 5 \ldots\ldots n}{1 \cdot 2 \cdot 3 \ldots\ldots \frac{n-1}{2}} \cdot 2^{\frac{n-1}{2}}.$$

(d). Since $(x + a)^n = (a + x)^n$ the developements of these two quantities must be equivalent also; now the co-efficient of x^{p-1} in the first series will be the p^{th} term of the series; and the same co-efficient in the second series will be the $\overline{n-p}^{\text{th}}$ term of that series; hence we deduce, that the co-efficients of the binomial theorem, equi-distant from each end, are equal. This might also have been deduced from the theory of complementary combinations, explained in Art. 129. (b).

(e). Sum of the co-efficients

$$= 1 + n + \frac{n(n-1)}{1 \cdot 2} + \&c. = (1 + 1)^n = 2^n.$$

(f). $$1 - n + \frac{n(n-1)}{1 \cdot 2} - \&c. = (1 - 1)^n = 0,$$

therefore, by transposing the negative terms, we find, sum of odd co-efficients = sum of even.

(g). Add the series in (e) and (f),

$$\therefore 2\left(1 + \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.\right) = 2^n;$$

or, sum of odd co-efficients = sum of even $= 2^{n-1}$.

(h). The general or p^{th} co-efficient of the binomial is

$$\frac{n(n-1)(n-2) \ldots\ldots \overline{n-p-2}}{1 \cdot 2 \cdot 3 \ldots\ldots (p-1)};$$

now, when n is an integer, it must be of the form, $2p$ or $2p + 1$, (See Art. 156.) hence one at least of the two first factors in the numerator must be divisible by 2; again, n must be of the form, $3p$, $3p + 1$ or $3p + 2$, therefore one at least of the three first factors must be divisible by 3, and so on; now there are as

N.B

many factors in the numerator as there are in the denominator, and since *each* factor in the numerator is divisible by one term in the denominator, therefore the product of all the factors in the numerator must be divisible by the product of all in the denominator; hence, when the index is an integer, every coefficient is an integer likewise.

140. By means of the binomial theorem, we may obtain a converging series for the extraction of the n^{th} root of a number which is not itself a complete n^{th} power.

Let N be the number, and let a^n be the n^{th} power nearest to it, so that, $N = a^n + x$, x being small compared with a^n.

$$\therefore N^{\frac{1}{n}} = (a^n + x)^{\frac{1}{n}} = a\left(1 + \frac{x}{a^n}\right)^{\frac{1}{n}}$$

$$= a\left(1 + \frac{1}{n}\frac{x}{a^n} + \frac{1-n}{n\,.\,2\,n}\frac{x^2}{a^{2n}} + \frac{(1-n)(1-2n)}{n\,.\,2\,n\,.\,3\,n}\frac{x^3}{a^{3n}} + \&c.\right).$$

The several terms being reduced to decimals and combined by addition or subtraction, as indicated by the signs, will give the value of the root to any required degree of accuracy.

141. We may, however, deduce a rule more practical than the foregoing, for approximating to the higher roots of irrational numbers; for if we suppose as before, a^n the nearest n^{th} power contained in N, b the remaining part of N, and x the correction due to a, which is supposed small in comparison with a^n, we shall have,

$$N = a^n + b; \text{ and } N^{\frac{1}{n}} = a + x,$$

$$\therefore a^n + b = (a + x)^n,$$

$$\therefore b = x\left(n\,a^{n-1} + \frac{n(n-1)}{1\,.\,2}\,a^{n-2}\,x + \&c.\right);$$

In order to obtain a first approximation, not much error will be introduced, by neglecting the terms within the brackets which are not multiplied by x, since x itself is supposed small; compared with a and its powers; hence,

$$b = n\,a^{n-1}\,x, \text{ and } \therefore x = \frac{b}{n\,a^{n-1}} \text{ nearly.}$$

For a second approximation, substitute this value of x in the *second* term of the series for b, this will necessarily lead to a more accurate result, since by this means, a term is taken into account which was entirely neglected in the first approximation; we shall have then,

$$b = x\left(n\,a^{n-1} \pm \frac{n\,(n-1)}{1\,.\,2}\,a^{n-2}\,\frac{b}{n\,a^{n-1}}\right) \text{ very nearly,}$$

$$\therefore\ x = \frac{2\,a\,b}{2\,n\,a^n \pm (n-1)\,b},$$

which is the quantity to be added to or subtracted from a, in order to obtain an approximate value of $N^{\frac{1}{n}}$.

(a). If $n = 2$, then

$$\sqrt{N} = \sqrt{a^2 \pm b} = a \pm \frac{2\,a\,b}{4\,a^2 \pm b} = a \pm \frac{2\,a\,b}{3\,a^2 + N}.$$

(b). If $n = 3$, then

$$\sqrt[3]{N} = \sqrt[3]{a^3 \pm b} = a \pm \frac{a\,b}{3\,a^3 \pm b} = a \pm \frac{a\,b}{2\,a^3 + N}.$$

142. *The sum of the squares of the co-efficients of a binomial, raised to the* n$^{\text{th}}$ *power, is equal to the middle co-efficient of the same binomial, raised to the power* 2 n.

Let a_1, a_2, a_3, &c. be the co-efficients of $(1 + x)^n$; then since $(1 + x)^{2n} = (1 + x)^n (x + 1)^n$, by developing both sides, we have,

$$1 + 2\,n\,x + \frac{2\,n(2\,n-1)}{1\,.\,2}\,x^2 + \ldots\ldots$$

$$+ \frac{2\,n\,(2\,n-1)\,(2\,n-2)\,\ldots\ldots\,(n+1)}{1\,.\,2\,.\,3\,\ldots\ldots\,n}\,x^n + \text{&c.}$$

$$= \left\{1 + a_1\,x + a_2\,x^2 \ldots\ldots + a_n\,x^n\right\}$$

$$\left\{x^n + a_1\,x^{n-1} + a_2\,x^{n-2} \ldots + a_{n-1}\,x + a_n\right\},$$

and equating the co-efficients of x^n on each side, we have,

that on the second side $= 1^2 + a_1^2 + a_2^2 + a_3^2 \ldots + a_n^2$,

that on the first side $= \dfrac{2\,n\,(2\,n-1)\,\ldots\ldots\,(n+1)}{1\,.\,2\,.\,3\,\ldots\ldots\,n}$,

or $1^2 + a_2^2 + a_3^2 \ldots\ldots + a_n^2 =$ middle co-efficient of $(1 + x)^n$.

(*a*). The co efficient $\frac{2n(2n-1)\ldots\ldots(n+1)}{1.2.3\ldots\ldots n}$ may also be written

$$\frac{2n(2n-1)\ldots.(n+1)\times n(n-1)\ldots.3.2.1}{1.2.3\ \ldots n\times n(n-1)\ldots.3.2.1}=\frac{1.2.3\ldots.2n}{(1.2.3\ldots.n)^2},$$

hence,

$$1^2+n^2+\left(n\frac{n-1}{1.2}\right)^2+\&c.=\frac{1.2.3\ldots.2n}{(1.2.3\ldots.n)^2}.$$

For examples in the binomial theorem, see *Peacock's Algebra*, *p.* 271, and *p.* 293, for various equivalent forms of the series for $(1+x)^n$. Also several examples may be found in *Fennel's Algebra*, *p.* 258, and *Hind's*, *p.* 248.

CHAP. XI.

ON THE THEORY OF NUMBERS.

SECT. I.

ON SURDS.

143. *In any equation consisting of rational quantities, and quadratic surds or impossible quantities, the rational parts on each side are equal, and also the irrational or impossible parts.*

For, let $a + \sqrt{\pm x} = b + \sqrt{\pm y}$ be such an equation; and if a be not $= b$; let $a + m = b$; then

$$m + \sqrt{\pm x} = \sqrt{\pm y}, \text{ and by squaring,}$$

$$m^2 + 2m\sqrt{\pm x} \pm x = \pm y;$$

$$\therefore \sqrt{\pm x} = \frac{\pm y \mp x - m^2}{2m}$$

a possible and rational quantity, which is in either case contrary to the supposition made with respect to the quantity $\sqrt{\pm x}$.

144. If a, b, c, &c. be the prime factors of a number N, it may be represented thus, $N = a^m b^n c^p$, &c.

First, if N be a prime; $b = 1$, $c = 1$, &c.; and $m = 1$, and in that case, $N = a$;

Secondly, if N be not a prime, divide it first by the highest power a^m of one of its prime factors, by which it is divisible, and the quotient by the highest power of b as b^n, &c.

Then ultimately, $N = a^m b^n c^p$, &c.

(*a*) Hence every square number is of the form $a^{2m} b^{2n} c^{2p}$, &c. and every cube of the form, $a^{3m} b^{3n} c^{3p}$, &c.

145. The following equations are impossible ones, when the surds introduced are irreducible.

(1) $\sqrt{x} = a + \sqrt{y}$;

For $\sqrt{x} = \sqrt{y}$, by Art. 143; and $\therefore a = 0$.

Hence the square root of a quantity cannot be partly rational and partly a quadratic surd.

(2) $\sqrt{xy} = a$; when x is prime to y;

For, then we should have $a^2 = xy$ which is an impossible form for a square number, a^2; Art. 144 (*a*).

Hence, if two quadratic surds cannot be reduced to others which have the same irrational part, their product cannot be rational.

(3) $\sqrt{x} = \sqrt{y} \pm \sqrt{z}$; or, $a = \sqrt{y} \pm \sqrt{z}$.

For, then, a^2 or $x = y \pm 2\sqrt{yz} + z$;

and $\sqrt{yz} = \pm \frac{1}{2}(x - y - z)$ or $\pm \frac{1}{2}(a^2 - y - z)$,

which is impossible by the last case.

Hence one quadratic surd cannot be made up of two others which have not the same irrational part.

(4) $\sqrt{x} = \frac{a}{b}$; where $\frac{a}{b}$, is in its lowest terms.

For, then $x = \frac{a^2}{b^2}$ a whole number equal to a fraction, since $\frac{a^2}{b^2}$ is in its lowest terms as well as $\frac{a}{b}$. See Art. 159 (*b*).

Hence the square root of an integer that is not a square cannot be expressed by any fraction; and therefore there is no fraction $\frac{1}{q}$; which will measure the surd $\sqrt{x}$ and unity, this is usually expressed by saying that $\sqrt{x}$ is *incommensurable* with unity.

Ex. If x be the side of a square, the diagonal $= x\sqrt{2}$, hence, the diagonal of a square is incommensurable with respect to one of its sides.

(4). $\sqrt{x} \pm \sqrt{y} = \sqrt{x'} \pm \sqrt{y'}$, when x and y are prime to each other;

For, squaring and transposing

$\sqrt{xy} \mp \sqrt{x'y'} = \pm \frac{1}{2}(x' + y' - x' - y')$, which is impossible, if $\sqrt{x'y'}$ be rational by (2); and by (1) if $\sqrt{x'y'}$ be irrational.

146. *If* n *be an even number,* $\sqrt{x}$, *a quadratic surd, and* $(a + \sqrt{x})^{\frac{1}{n}} = \sqrt{y} + \sqrt{z}$; *where* $\sqrt{y}$, *and* $\sqrt{z}$ *may each be rational or irrational; then will* $(a - \sqrt{x})^{\frac{1}{n}} = \sqrt{y} - \sqrt{z}$.

For, $a + \sqrt{x} = (\sqrt{y} + \sqrt{z})^n$

$$= y^{\frac{n}{2}} + n y^{\frac{n-1}{2}} \sqrt{z} + \frac{n(n-1)}{1.2} y^{\frac{n-2}{2}} z + \&c.$$

Now, since n is divisible by 2, all the odd terms in this series will be rational, whilst all the even terms will involve an irrational quantity; hence equating rational and irrational parts, Art. 143.

$$a = y^{\frac{n}{2}} + \frac{n(n-1)}{1.2} y^{\frac{n-2}{2}} z + \&c.$$

and, $\sqrt{x} = n y^{\frac{n-1}{2}} \sqrt{z} + \frac{n(n-1)(n-2)}{1.2.3} y^{\frac{n-3}{2}} z \sqrt{z} + \&c.$

$$\therefore a - \sqrt{x} = y^{\frac{n}{2}} - n y^{\frac{n-1}{2}} \sqrt{z} + \frac{n(n-1)}{1.2} y^{\frac{n-2}{2}} z - \&c.$$

$$= (\sqrt{y} - \sqrt{z})^n$$

$$\therefore (a - \sqrt{x})^{\frac{1}{n}} = \sqrt{y} - \sqrt{z}.$$

147. *If* n *be an odd number, and* $(p\sqrt{x} + q\sqrt{y})^{\frac{1}{n}} = \sqrt{x} + \sqrt{y}$, *where* $\sqrt{x}$ *and* $\sqrt{y}$ *may each be rational or irrational; then also, will* $(p\sqrt{x} - q\sqrt{y})^{\frac{1}{n}} = \sqrt{x} - \sqrt{y}$.

For, as before, $p\sqrt{x} + q\sqrt{y} = (\sqrt{x} + \sqrt{y})^n$

$$= x^{\frac{n}{2}} + n x^{\frac{n-1}{2}}\sqrt{y} + \frac{n(n-1)}{1.2} x^{\frac{n-2}{2}} y + \&c.$$

Now, since n is *not* divisible by 2, each of the odd terms of this series will involve $\sqrt{x}$, and each of the even ones $\sqrt{y}$. Also no part of $p\sqrt{x}$ can be composed of the surds of y. Art. 144. (3) hence

$$p\sqrt{x} = x^{\frac{n}{2}} + \frac{n(n-1)}{1.2} x^{\frac{n-2}{2}} y + \&c. \text{ and similarly,}$$

$$q\sqrt{y} = n x^{\frac{n-1}{2}}\sqrt{y} + \frac{n(n-1)(n-2)}{1.2.3} x^{\frac{n-3}{2}} y\sqrt{y} + \&c. \text{ hence}$$

$$p\sqrt{x} - q\sqrt{y} = x^{\frac{n}{2}} - n x^{\frac{n-1}{2}}\sqrt{y} + \frac{n(n-1)}{1.2} x^{\frac{n-2}{2}} y - \&c.$$

$$= (\sqrt{x} - \sqrt{y})^n$$

$$\therefore (p\sqrt{x} - q\sqrt{y})^{\frac{1}{n}} = \sqrt{x} - \sqrt{y}.$$

148. *To extract the square root of a binomial, one of whose factors is a quadratic surd, and the other rational.*

Let $a \pm \sqrt{b}$ be the binomial; and assume,

$$\sqrt{a \pm \sqrt{b}} = \sqrt{x} \pm \sqrt{y}; \text{ hence, squaring,}$$

$$a \pm \sqrt{b} = x \pm 2\sqrt{xy} + y,$$

$$\left.\begin{aligned} \therefore a &= x + y \\ \sqrt{b} &= 2\sqrt{xy} \end{aligned}\right\} \text{ by equating rational and irrational parts;}$$

squaring, these two equations, $a^2 = x^2 + 2xy + y^2$;
$b = 4xy$

subtract these last equations,

$$\therefore a^2 - b = x^2 - 2xy + y^2 = (x-y)^2.$$

hence we have, $x - y = \sqrt{a^2 - b}$

and, from above, $x + y = a$;

$$\therefore x = \frac{a + \sqrt{a^2 - b}}{2}; \text{ and } y = \frac{a - \sqrt{a^2 - b}}{2};$$

$$\therefore \sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

(*a*). It appears from this result that the root of $a \pm \sqrt{b}$, cannot be exhibited in a simple form, unless the root of $a^2 - b$ can be extracted; that is, unless $a^2 - b$ be a complete square.

(*b*). We may apply the same principle to extract the root of such a quantity as $\sqrt{a^2 c} \pm \sqrt{b}$ by putting it under the form $\sqrt{c}\left\{a \pm \sqrt{\frac{b}{c}}\right\}$ and then extracting, as before, the root of the factor $\left\{a \pm \sqrt{\frac{b}{c}}\right\}$ and multiplying that root by $\sqrt[4]{c}$.

(*c*.) Also frequently to a trinomial quantity, by assuming its root $= \sqrt{x} \pm \sqrt{y} \pm \sqrt{z}$. See "*Hind's Algebra*," p. 210.

For examples in the preceding rule see *Wood's Algebra*, p. 130. Also *Bridge's*, p. 175.

149. *To extract the* n$^{\text{th}}$ *root of a binomial, one or both of whose factors are possible quadratic surds.*

Let the binomial be $\sqrt{a} \pm \sqrt{b}$; and assume,

$\sqrt[n]{g(\sqrt{a} \pm \sqrt{b})} = \sqrt{x} \pm \sqrt{y}$; g being arbitrary;

$\therefore \sqrt[n]{g(\sqrt{a} \mp \sqrt{b})} = \sqrt{x} \mp \sqrt{y}$; Art. 147. Multiply these equations together.

$$\therefore \sqrt[n]{g^2(a-b)} = x - y,$$

Let g be so assumed, that $g^2(a-b)$ may be a perfect n^{th} power $= k^n$, suppose; then,

$x - y = k$, and from the two first equations,

$$x \pm 2\sqrt{xy} + y = \sqrt[n]{g^2(\sqrt{a} \pm \sqrt{b})^2}$$

$$x \mp 2\sqrt{xy} + y = \sqrt[n]{g^2(\sqrt{a} \mp \sqrt{b})^2}, \text{ therefore by addition,}$$

$$2(x+y) = \sqrt[n]{g^2(\sqrt{a} \pm \sqrt{b})^2} + \sqrt[n]{g^2(\sqrt{a} \mp \sqrt{b})^2}$$

Now, $2(x+y)$ is always an integer, when the root can be exhibited in the form of a binomial surd; hence, if we assume

$s =$ the integer next *greater* than the true value of $\sqrt[n]{g^2(\sqrt{a} \pm \sqrt{b})^2}$

and $t =$ ..... *less* ..... of $\sqrt[n]{g^2(\sqrt{a} \mp \sqrt{b})^2}$

since the sum of these two surds is an integer, the fractional parts must destroy each other; and we shall have,

$2(x+y) = s + t$, exactly;

also, $x - y = k$; from which two equations we get,

$$x = \frac{s+t+2k}{4}; \text{ and, } y = \frac{s+t-2k}{4};$$

Hence, the n^{th} root required $= \dfrac{\sqrt{x} \pm \sqrt{y}}{\sqrt{g}}$

$$= \frac{1}{2\sqrt[n]{g}} \left\{ \sqrt{s+t+2k} \pm \sqrt{s+t-2k} \right\}.$$

For examples, see Wood's Algebra, p. 132.

(*a*). The quantity $g^2(a-b)$ may always be made a perfect n^{th} power, by assuming $g^2 = (a-b)^{n-1}$; but sometimes a less value of g, which will answer the purpose may be found, by decomposing $a - b$ into its prime factors: (Art. 144): Let these be

$\alpha^{n-p}.\ \beta^{n-q}.\ \gamma^{n-r}.$ &c.; then, if g^2 be assumed $= a^p \beta^q \gamma^r$ &c. we shall have

$g^2 (a - b) = \alpha^n \beta^n \gamma^n$, &c. which is of the required form.

150. *To extract the* n$^{\text{th}}$ *root of a binomial, one of whose factors is rational, and the other either irrational or impossible, by the solution of an equation of* n *dimensions.*

Let $a \pm \sqrt{b}$ be the given binomial; and assume,

$$\sqrt[n]{g\,(a \pm \sqrt{b}} = x \pm \sqrt{y} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\therefore\ \sqrt[n]{g\,(a \mp \sqrt{b})} = x \mp \sqrt{y},\ \text{Art. 146.}$$

$$\text{and}\ \sqrt[n]{g^2 (a^2 - b)} = x^2 - y\ ;$$

Let g be so assumed that $g^2 (a - b)$ may be a perfect n^{th} power $= k^n$, suppose,

$$\therefore\ x^2 - y = k;\ \text{whence,}\ y = x^2 - k\ \ldots\ \ldots\ldots\ldots (2)$$

Again, from (1);

$$g\,a \pm g\sqrt{b} = x^n \pm n\,x^{n-1}\sqrt{y} + \frac{n\,(n-1)}{1\,.\,2} x^{n-2} y \pm \&c.$$

And equating rational parts,

$$g\,a = x^n + \frac{n\,(n-1)}{1\,.\,2} x^{n-2} y + \&c.\ \text{substitute from (2),}$$

$$= x^n + \frac{n\,(n-1)}{1\,.\,2} x^{n-2} (x^2 - k) + \&c.$$

This is an equation of n dimensions, from the solution of which may be derived a value of x, and then y from (2).

Ex. 1. *To extract the square root of* $-4 + 2\sqrt{-5}$.

Assume, $\sqrt{g\,(-4 + 2\sqrt{-5}} = x + \sqrt{y}$;

$$\therefore\ \sqrt{g\,(-4 - 2\sqrt{-5})} = x - \sqrt{y}$$

and, $\sqrt{g^2\,(16 + 20)} = x^2 - y$, here evidently, $g^2 = 1$,

and we have $x^2 - y = \sqrt{36} = 6$.

Again, by squaring the first equation, observing that $g^2 = 1$, we have

$$-4 + 2\sqrt{-5} = x^2 + 2x\sqrt{y} + y,$$

whence, $-4 = x^2 + y$

But, $6 = x^2 - y$

and by addition, $2 = 2x^2$; and $x = 1$,

by subtraction, $-10 = 2y$; and $y = -5$

and the root required $= x + \sqrt{y} = 1 + \sqrt{-5}$.

Ex. 2. *To extract the square root of* $2\sqrt{-1}$.

Let $\sqrt{g(0 + 2\sqrt{-1})} = x + \sqrt{y}$;

$\therefore \sqrt{g(0 - 2\sqrt{-1})} = x - \sqrt{y}$,

and $\sqrt{g^2(0 + 4)} = x^2 - y$; here also $g = 1$

and $x^2 - y = 2$; also squaring, and equating rational parts in the first equation.

$x^2 + y = 0$; from which two equations, $x = 1$, and $y = -1$.

Hence, the root required $= 1 + \sqrt{-1}$.

CHAP. XI. SECTION II.

ON THE DIFFERENT SCALES OF NOTATION.

151. We have seen (Art. 59) that any number (N) may be thus expressed,

$$N = a_n 10^n + a_{n-1} 10^{n-1} \ldots\ldots a_1 10 + a_0$$

where the value of the digits increase each time by a power of 10; the number 10 is in this case called *the radix of the scale of notation* in which the number is expressed; any other number r may be chosen for this purpose and then N will be expressed by

$$b_m r^m + b_{m-1} r^{m-1} + \&c. \ldots\ldots + b_1 r + b_0.$$

152. *Having given a number* (N) *expressed to a radix* r *to find its digits when expressed to a radix* ρ.

Suppose that to radix ρ,

$$N = c_p \rho^p + c_{p-1} \rho^{p-1} + \ldots\ldots c_1 \rho + c_0.$$

Now, we may observe, that if we divide N, the given number, by the new radix ρ, the remainder will be c_0; if we divide it a second, third, p^{th}, &c. time, the remainders will be c_1, c_2, &c. c_p. So that to obtain the digits required, we must "divide the given number by the new radix, the remainders taken in a reverse order will be the new digits."

(a). The division of the number N by the new radix ρ must take place in the scale r in which it is first expressed; hence if we have to transform a given number (radix r) to the common or denary scale, it is best done by simple addition; thus 234 in the *quinary* scale $= 2 . 5^2 + 3 . 5 + 4 = 69$, in common notation.

(*b*). And in transforming a number from radix r to radix ρ, it is best to transform it first, from the scale of r to the denary, (by addition) and then from the denary to that of ρ (by division, as in Art. 152.) This avoids the necessity of carrying on the operation of division in an unusual scale.

(*c*). If N be the same number expressed in the scales r and ρ, we have,

$b_m r^m + b_{m-1} r^{m-1} + \ldots . b_1 r + b_0 = N = c_p \rho^p + c_{p-1} \rho^{p-1} + \ldots . c_1 \rho + c_0$

so that of the four sets of quantities, the radixes of each scale, and the set of digits in each scale, any three being given the other may be found. If, however, we have given the two sets of digits and the radix r, to find the radix ρ, we shall have to solve an equation of p dimensions.

153. *To find the condition of divisibility of one number* (N) *by another* (D).

Let $N = b_m r^m + b_{m-1} r^{m-1} + \ldots . b_1 r + b_0$

$= b_m(D + \overline{r - D})^m + b_{m-1}(D + \overline{r - D})^{m-1} + \ldots .$

$\ldots . + b_1(D + r - D) + b_0.$

Now it will be evident by expanding each of the quantities

$b_m(D + \overline{r - D})^m$; $b_{m-1}(D + \overline{r - D})^{m-1}$ &c.

that they will each be divisible by D, provided $b_m(r - D)^m$, $b_{m-1}(r - D)^{m-1}$, &c. be so; hence $\frac{N}{D}$ will be a whole number, if,

$b_0 + b_1(r - D) +$ &c. $\ldots . + b_{m-1}(r - D)^{m-1} + b_m(r - D)_m$

be divisible by D.

Ex. Let $D = 13$; $r = 10$; then $r - D = -3$, and the criterion of divisibility of any number by 13, is the fact of

$b_0 - 3 b_1 + 3^2 b_2 - 3^3 b_3 +$ &c. being so.

(*a*). If $r = 10$, and $D = 9$, it appears that a number is divisible by 9, if the sum of its digits be divisible by 9; hence if *any* number and the sum of its digits be divided by 9, the remainders in each case will be equal.

(*b.*) Let N and N' be two numbers, and let the number 9 be contained in N, p times with a remainder R, and in N', p' times, with a remainder R'; then we have,

$$N = p\,9 + R, \text{ and } N' = p'\,9 + R';$$

and by multiplying these together,

$$NN' = pp'\,9^2 + p'\,R\,9 + p\,R'\,9 + RR';$$

hence we have a test to determine the accuracy of multiplication; for if two numbers (N and N') which are to be multiplied together, be each divided by the number 9, the remainders (R and R') when multiplied together ought to be the same as the remainder (RR') arising from the division of the product (NN') by the same quantity 9; provided there has been no mistake in the operation. It will, however, suffice instead of actually dividing the numbers (N', N', and NN') themselves, to divide *the sum of their digits* by 9; as in that case the effect (as far as the remainders are concerned) will be the same as if those quantities themselves were divided by 9; by (*a*). This is called "*Casting out the nines.*"

(*c*). The above is not an unerring test of the accuracy of multiplication, since it is evident that it will not detect an error, provided that error be, in defect or excess, a multiple of 9.

154. The operations of addition, multiplication, &c. are carried on in the scale of which the radix is r, in the same manner as in the common scale, observing that as in the one case 1 is carried to, or borrowed from, the next superior unit for every 10 units, so in the other case, 1 must be carried, or borrowed, for every r units.

155. Any even number may be represented by $2n$, and any odd one by $2n \pm 1$; hence the sum or difference of two odd numbers, $2n+1$ and $2n'+1$, may be represented by $2(n+n'+1)$ or by $2(n+n')$ each of which being divisible by 2 is an even number. Similarly it may be shewn that

(1) The sum or difference of two even numbers is even:

(2) The sum of any number of even numbers is even:

(3) Any even number of odd numbers is even, but any odd number of odd numbers is odd:

(4) The sum of an even and odd number is odd:

(5) The product of any number of factors one of which is even will be even, but the product of any number of odd numbers is odd:

(6) Hence every power of an even number is even, and every power of an odd number is odd:

(7) Hence also the sum or difference of any power and its root is even; for, the power and root will be either both even or both odd, and in either case the sum or difference is even by (1).

156. *If an odd number divide an even number it will also divide the half of it.*

Let $A = 2n$; $B = 2n' + 1$, be an even and odd number, such that B is a divisor of A, let the division be made, and call the quotient q, then we have

$2n = q(2n' + 1)$, hence q is even or of the form $2n''$;

$$\therefore 2n = 2n''(2n' + 1) \text{ and } n'' = \frac{n}{2n' + 1},$$

that is, $n = \frac{1}{2}A$ is divisible by B if A itself be so.

CHAP. XI. SECTION III.

ON PRIME NUMBERS.

157. *Every number may be expressed by one of the forms*

$$p\,n,\ p\,n \pm 1,\ p\,n \pm 2,\ \&c.\ to\ p\,n \pm \frac{p}{2}\ or\ p\,n \pm \frac{p+1}{2}$$

according as p is even or odd.

For every number is either divisible by p, (in which case it is of the form $p\,n$) or it is not, in the latter case the remainder must be one of the numbers 1, 2, 3, &c. $\overline{p-1}$; hence every number is of the form $p\,n$, or $p\,n+1$, or $p\,n+2$, &c. or, $p\,n+\overline{p-1}$;

Now, $p\,n+\overline{p-1} = p(n+1)-1 = p\,n'-1$;
putting n' for $n+1$.

Similarly, $p\,n+\overline{p-2} = p(n+1)-2 = p\,n'-2$;
hence by using the double sign $\pm$ we can comprise all the forms in the series $p\,n,\ p\,n \pm 1,\ \ldots\ldots\ p\,n+\overline{p-1}$,
by the series,

$$p\,n,\ p\,n \pm 1\ \&c.\ up\ to,\ p\,n \pm \frac{p}{2},\ or\ p\,n \pm \frac{p+1}{2}.$$

(*a*). The number p is called the *modulus* of the form.

(*b*). To an even modulus prime numbers must be of the form $p\,n \pm 1$, $p\,n \pm 3$, &c. since the intermediate numbers in the series would be divisible by 2.

Ex. Every prime number is of one of the forms $6n \pm 1$, for here the form $6\,n \pm 3$ is excluded, as being divisible by 3.

(*c*.) When a number P is expressed by the form $p\,n \pm r$, r must necessarily be less than p; and if P be prime to its modulus p, r must also be prime to p; for $P = p\,n \pm r$, and if

r and p had a common measure, $pn \pm r$ (or P,) would be divisible by that common measure, Art. 53, and in that case P would not be prime to p, which is contrary to the supposition.

(d). The converse of (b) is not true, viz. that all numbers contained in those forms are necessarily primes. See Art. 163.

thus if $n = 4$, $6n+1 = 25$ wh. is not prime.

158. *If* a *and* b *be any two numbers prime to each other, then* a $\pm$ b *is prime to each of them.*

For if $a \pm b$ and b had any common divisor, then $(a \pm b) \mp b$ and b, or a and b would have a common divisor, which is contrary to the hypothesis.

(a). Conversely, if a number consist of two parts, and be prime to one of those parts, it will be prime to the other.

(b). Hence also $a+b$ is prime to $a - b$, or they have only the common measure 2; for if they have any other common measure, their sum and difference 2 a and 2 b, will have the same common measure, but since a and b are prime to each other, 2 a and 2 b can only have the common measure 2.

It is evident that $a + b$, and $a - b$, will only have the common measure 2, when a and b are both odd.

159. *If a number* (c) *be prime to each of two others* (a) *and* (b), *it is prime to their product* (a b).

If possible, let d be a prime divisor of both c and $a\,b$; hence by hypothesis it cannot be a divisor of either a or b since c is prime to both a and b.

Suppose $\frac{a\,b}{d} = s$, $\therefore \frac{a}{d} = \frac{s}{b}$,

and since d is a prime, and not a divisor of a, $\therefore \frac{a}{d}$ is a fraction in its lowest terms.

If $\frac{s}{b}$ be not a fraction in its lowest terms, let d' be its greatest common measure; so that, $s = d' r$; $b = d' r'$;

$$\therefore \frac{a}{d} = \frac{s}{b} = \frac{d' r}{d' r'} = \frac{r}{r'};$$

The first and last of these fractions must be identical, as they are both irreducible;

hence, $a = r$, and $d = r'$;

also, $b = d' r' = d d'$; and $\therefore \frac{b}{d} = d'$;

a whole number, which is absurd; since d has been shewn not to be a divisor of b.

(*a*). Hence if c be prime to a, it is prime to the product of a and a, or a^2, also to a^3 and generally to a^n.

(*b*). Since a^n is prime to c, if a be prime to c; therefore a^n is prime to the product of c and c, or c^2; and generally to c^n; that is, if $\frac{a}{c}$ be a fraction in its lowest terms, $\frac{a^n}{c^n}$ is in its lowest terms likewise.

(*c*). Hence also if c be prime to x, y, z, &c. separately, it is prime to their product $x \,.\, y \,.\, z$. &c.

(*d*). If c divide the product $a\ b$, but is prime to one of its factors, it must be a divisor of the other.

160. *If* $\frac{a}{b}$ *be an irreducible fraction, the terms of every other fraction,* $\left(\frac{A}{B}\right)$ *which is equal to* $\frac{a}{b}$, *must be severally equimultiples of* a *and* b.

For, if $\frac{a}{b} = \frac{A}{B}$, $\therefore A = \frac{aB}{b}$; which last is therefore a whole number,

$\therefore a\ B$ must be divisible by b;

but a is not divisible by b, by hypothesis;

therefore B must be divisible by b, or in other words, B must be a multiple of b; and similarly, A must be a multiple of a.

161. The p^{th} co-efficient of a binomial raised to the n^{th} power is (Art. 139 (h).)

$$^nC_p = \frac{n\,(n-1)\,(n-2)\,\ldots\,(n-p+1)}{1\,.\,2\,.\,3\,\ldots\,p};$$

$$\therefore 1\,.\,2\,.\,3\,\ldots\,p \times {}^nC_p = n\,(n-1)\,(n-2)\,\ldots\,(n-p+1).$$

Since the second side is a multiple of n, the first must be so likewise, and if n be a prime and be $> p$, then ^nC_p must be divisible by n, hence in that case all the terms of the bionomial $(x+a)^n$ are divisible by n except x^n and a^n. Similarly it may be proved that under similar circumstances all the terms of $(a+b+c+\ldots)^n$ are divisible by n except $a^n+b^n+c^n+$ &c.

162. *To ascertain whether a given number be a prime number or not.*

Divide it by all the prime numbers which are equal to or less than the square root of itself, if it be not divisible by any one of these, it is a prime.

For, every number N that is not a prime may be represented by $N = a\,b$, Art. 144. where N is divisible both by a and b.

Now if $a = b$, they are each equal to $\sqrt{N}$, N being in this case a square number. Again, if $a > \sqrt{N}$ then b must be $< \sqrt{N}$, otherwise we should have $a\,b > N$, which is contrary to the supposition; conversely if $b > \sqrt{N}$, then we have $a < \sqrt{N}$; in either case, we have the number N divisible by a number less than the square root of itself; and this is true of all numbers of the form $N = a\,b$; that is, of all numbers not primes; hence if a number cannot be so divided, it is a prime.

163. *No single Algebraical formula can be found which contains prime numbers only.*

It may be proved that every Algebraical formula may be put under the form

$$P = a_1 x + a_2 x^2 + a_3 x^3 + \&c.$$

If possible, let this contain prime numbers only; and let P become a determinate prime number (P_m) when x becomes $= m$:

$$\text{then, } P_m = a_1 m + a_2 m^2 + a_3 m^3 + \&c.$$

Now in the first formula, if we make $x = m + p P_m$ the result ought still to be a prime number, we have then,

$$P = \left\{ \begin{array}{l} a_1 x \ldots\ldots = a_1 (m + p P_m) \\ + a_2 x^2 \ldots\ldots = a_2 (m + p P_m)^2 \\ + a_3 x^3 \ldots\ldots = a_3 (m + p P_m)^3 \\ + \&c. \ldots\ldots = \&c. \end{array} \right\}$$

$= a_1 m + a_2 m^2 + a_3 m^3 + \&c.$ + other terms involving P_m, P^2_m, &c.
$= P_m$ + remaining terms multiplied by P_m, $P_m{}^2$, &c. which is divisible by P_m, and consequently cannot be a prime; or in other words, such a value may be given to the indeterminate quantity in every Algebraical formula, as will render it divisible by some other number.

(a). There are some remarkable formulæ which contain a great number of primes, if x equal any term of the series, 0, 1, 2, 3, &c. then

the first 40 terms of, $x^2 + x + 41$ are primes.

..... 29 ..... $2x^2 + 29$ .....

..... 17 ..... $x^2 + x + 17$ .....

and if x be any term of the series 1, 2, 2^2, 2^3, &c. the first 31 terms of $2^x + 1$ are primes.

164. *Neither the sum nor the difference of two fractions which are in their lowest terms, and of which the denominator of the one contains a factor not contained in the other, can be an integer.*

For let, $\frac{a}{A}$ and $\frac{b}{Bt}$ be two fractions in their lowest terms, of which t, a factor in the denominator of one, is not contained in A, the denominator of the other. Then if $\frac{a}{A} \pm \frac{b}{Bt}$ be an integer, $\frac{aBt \pm Ab}{ABt}$ must be one likewise; which cannot be the case unless Ab be divisible by ABt, that is, unless Ab be divisible by each of the factors of ABt; but A and b are each prime to t, hence Ab cannot be divisible by t; Art. 158; and therefore $\frac{a}{A} \pm \frac{b}{Bt}$ cannot be an integer.

(a). Hence if the denominators of two fractions in their lowest terms be prime to each other the sum or difference of these fractions cannot be an integer.

(b). And if two fractions are in their lowest terms, their product is in its lowest terms.

165. *The number of prime numbers is infinite.*

If not, let their number $= n$, and let the greatest $= P$; then the continued product $1 . 2 . 3 . 4 . 5 \ldots\ldots P$ will be divisible by every term of the series $1, 2, 3 \ldots\ldots P$; and $1 . 2 . 3 \ldots\ldots P + 1$ will be divisible by no term of that series; hence if this last formula be divisible by *any* prime number it must be by one greater than P, or else it must be itself a prime, and in either case we have found a prime greater than the greatest prime, which is absurd; and the same may be shewn of any finite values of n and P.

166. *If* a *and* b *be any two numbers prime to each other, and each of the terms of the series*

$$\mathrm{b},\ 2\ \mathrm{b},\ 3\ \mathrm{b},\ 4\ \mathrm{b},\ \ldots\ldots\ \overline{\mathrm{a}-1}\ .\ \mathrm{b},$$

be divided by a, *they will each leave a different positive remainder.*

For if two of these terms when divided by a leave the same remainder, let these two terms be represented by $x\,b$ and $y\,b$; then, $x\,b - y\,b$ or $b\,(x - y)$ is divisible by a, which is absurd, since a is prime to b, and $x - y$ less than a. Art. 159 (d).

(a). Since all the above remainders are different from each other, and $a - 1$ in number, therefore these remainders include all numbers from 1 up to $a - 1$.

(b). Hence, some one of these remainders will $= 1$; that is, a number x, less than a, may always be found that will render $b\,x - 1$ divisible by a, or the equation $b\,x - a\,y = 1$, is always possible, if a and b are prime to each other.

(c). The same is true of the equation $b\,x - a\,y = -1$ for since $a - 1$ must be one of the remainders, a number x may be found which renders $b\,x - (a - 1)$ divisible by a; that is, the equation $b\,x - a\,y = a - 1$ is always possible, or

$$b\,x - a\,(y + 1) = -1$$

is always possible, or

$$b\,x - a\,y' = -1\ ;\ \text{by making } y + 1 = y'.$$

FERMAT'S THEOREM.

167. *If* n *be a prime number and* h *a number not divisible by* n, *then* $\mathrm{h}^{n-1} - 1$ *is divisible by* n.

Since all the terms of the expanded polynomial

$$(a + b + c + \&c.)^n$$

are divisible by n, except $a^n + b^n + c^n + \&c.$, Art. 160; therefore $(a + b + c + \&c.)^n$ may be thus represented,

$$(a + b + c + \&c.)^n = a^n + b^n + c^n + \&c. + n\,K.$$

Let now, $a = b = c = \&c. = 1$, and let h be the number of terms of the polynomial; then,

$$h^n = h + n\,K \text{ or } n\,K = h^n - h = h(h^{n-1} - 1);$$

and since $n\,K$ is divisible by n therefore $h\,(h^{n-1} - 1)$ must be so likewise; but n is prime to h by hypothesis, therefore $h^{n-1} - 1$ is divisible by n. Art. 158 (*d*).

(*a*). n being a prime number, $n - 1$ must necessarily be an even number, as $2\,q$; hence we may assume

$$h^{n-1} - 1 = (h^q + 1)\,(h^q - 1),$$

and it appears from the above, that n must divide one or other of these factors; that is, the remainder from the division of h^q by n is ± 1, when n is a prime number > 2, and $q = \frac{n-1}{2}$.

(*b*). Since $h^n - h$ is divisible by n, therefore h^n and h leave the same remainder when divided by n.

(*c*). Also, since h is a prime number, each of the terms of the series 1, 2, 3, 4, &c. $\overline{n-1}$, are prime to n; hence the theorem will hold good, when any one of these numbers are substituted for h.

For more theorems deduced from the above, see *Ency. Metropolitana, Art.* "*Theory of Numbers,*" Art. 59. Also *Hind's Algebra, p.* 413.

SIR JOHN WILSON'S THEOREM.

168. *If* n *be a prime number then will*

$$1.2.3.4.\ldots\ldots(n-1)+1$$

be divisible by n.

For, it is shewn by the Calculus of Finite Differences, that

$$1.2.3\ldots.n = n^n - \frac{n}{1}(n-1)^n + \frac{n\,(n-1)}{1.2}(n-2)^n \&c.$$

For n put $n-1$,

$$\therefore 1.2.3.....\overline{n-1}$$
$$= (n-1)^{n-1} - \frac{n-1}{1}(n-2)^{n-1} + \frac{(n-1)(n-2)}{1.2}(n-3)^{n-1} - \&c.$$

But since n is prime to $n-1$, $n-2$, &c., each of these quantities raised to the power $n-1$ will be of the form

$$nK_1 + 1, nK_2 + 1, nK_3 + 1, \&c. \quad \text{Art. 167.}$$

$$\therefore 1.2.3 ..... \overline{n-1}$$
$$= nK_1 + 1 - \frac{n-1}{1}(nK_2+1) + \frac{(n-1)(n-2)}{1.2}(nK_3+1) - \&c.$$
$$= 1 - \frac{n-1}{1} + \frac{(n-1)(n-2)}{1.2} - \&c. + n\left(K_1 - \frac{n-1}{1}K_2 + \&c.\right)$$

which, since $n-1$ is an even number, may be represented by

$$(1-1)^n - 1 + nQ = -1 + nQ;$$

$\therefore 1.2.3 ..... \overline{n-1} + 1 = nQ$, and is therefore divisible by n.

(a). The product $1.2.3.4 .... \overline{n-1}$ is the same as

$$1(n-1)\,2(n-2)\,3(n-3)\ \&c.\ \left(\frac{n-1}{2}\right)^2;$$

and this product with regard to its remainders, when divided by n, is the same as

$$\pm 1^2.2^2.3^2.4^2 .... \left(\frac{n-1}{2}\right)^2$$

the + or − sign being used according as $n-1$ is even or odd; that is, according as n is of the form $4n'+1$ or $4n'-1$; Art. 156;

$$\therefore \left(1.2.3....\frac{n-1}{2}\right)^2 + 1$$

is of the form of n' when n' is of the form $4n+1$,

$$\text{and } \left(1.2.3....\frac{n-1}{2}\right)^2 - 1$$

is of the form of n' when n' is of the form $4n-1$;

hence, it follows that every prime number of the form of $4n+1$ is a divisor of the sum of two squares.

(*b*). The latter form may be resolved into the two factors

$$\left\{\left(1.2.3....\frac{n-1}{2}\right)+1\right\}\quad\left\{\left(1.2.3....\frac{n-1}{2}\right)-1\right\}$$

which product being divisible by n, it follows that n is a divisor of one or other of these factors when it is a prime number of the form $4n'-1$.

(*c*). From the theorem demonstrated above, a number of others may be derived as

$$\frac{1^2.2^2.3.4.5....(n-3)(n-1)+1}{n}=e, \text{ an integer,}$$

$$\frac{1^2.2^2.3^2.4.5....(n-4)(n-1)+1}{n}=e, \text{ an integer,}$$

and so on, until we arrive at the same form as that in (*a*).

CHAP. XI. SECTION IV.

ON THE DIVISORS OF COMPOSITE NUMBERS.

169. *To find the number of divisors of any number* (N).

Let N be resolved into its prime factors $a^m\ b^n\ c^p\ d^q$ &c.

Then each term of the several series $a, a^2, a^3 \ldots\ldots a^m$;

$b, b^2, b^3 \ldots\ldots b^n$; $c, c^2, c^3 \ldots\ldots c^p$; &c.

will evidently be a divisor of N, and also every possible combination of them; that is, N will be divisible by every term of the continued product

$(1 + a + a^2 \ldots . a^m)\ (1 + b + b^2 \ldots . b^n)\ (1 + c + c^2 \ldots . c^p)$ &c. Art. 137.

The number of terms in this product is

$$(1 + m)\ (1 + n)\ (1 + p)\ \&c.$$

(a). Sum of all the divisors

$$= (1 + a + a^2 \ldots . a^m)\ (1 + b + b^2 \ldots . b^n)\ (1 + c + c^2 \ldots . c^p)\ \&c.$$

$$= \frac{a^{m+1} - 1}{a - 1} \times \frac{b^{n+1} - 1}{b - 1} \times \frac{c^{p+1} - 1}{c - 1} \times \&c. \quad \text{Art. 120.}$$

(b). In this result both unity and N are reckoned amongst the number of the divisors.

Ex. $360 = 2^3 . 3^2 . 5'$, hence the number of divisors of

$$360 \text{ is } = (1 + 3)\ (1 + 2)\ (1 + 1) = 24.$$

(c). The number of ways in which N can be divided into *two* factors will

$$= \tfrac{1}{2}(1 + m)\ (1 + n)\ (1 + p)\ \&c.$$

(*d*). If each of the two factors be prime to each other, they will consist only of the prime factors, *a*, *b*, *c*, &c. composing *N*, and their number is therefore

$$= \frac{(1+1)\ (1+1)\ (1+1)\ \&c.}{2};$$

if the number of prime factors, *a*, *b*, *c*, &c. $= k$, then 2^{k-1} will be the number of ways in which *N* may be resolved into two factors prime to each other.

Ex. $360 = 2^3 \cdot 3^2 \cdot 5^1$ has 24 divisors, hence it may be resolved into factors 12 different ways, and into factors prime to each other $2^{3-1} = 4$, different ways.

(*e*). Hence may be found a number *N'* having a given number (*w*) of divisors; for, resolve *w* into its prime factors *m'*, *n'*, *p'*, &c. and take,

$$N' = a^{m'-1} \cdot b^{n'-1} \cdot c^{p'-1},$$

where the values of *a*, *b*, *c*, &c. are arbitrary; then *N'* is a number having $m' \cdot n' \cdot p' \cdot$ &c. $= w$, divisors, as appears from what has been demonstrated above.

170. *If* $N = a^m b^n c^p$ &c. *then the number of integers prime to and less than* N, *will*

$$= N \frac{a-1}{a} \times \frac{b-1}{b} \times \frac{c-1}{c} \&c.$$

First, let $n = p =$ &c. $= 0$, and $\therefore N = a^m$.

Then of the numbers 1, 2, 3, 4 .... up to a^m, it is evident that there will be one, at intervals of every (*a*) terms, which will be a multiple of *a*, and therefore not prime to *N*, these will be *a*, 2 *a*, 3 *a*, &c. $a^{m-1} \cdot a$, and their number will $= a^{m-1}$; hence the number of the remaining terms of that series will be

$$a^m - a^{m-1} = a^{m-1}(a-1) = N\frac{a-1}{a}.$$

Again, if $N = a^m b^n$, we shall have in a similar manner

$a^{m-1} b^n$ terms divisible by *a*,
$a^m b^{n-1}$ terms divisible by *b*,
$a^{m-1} b^{n-1}$ terms divisible by *a b*.

But as the first expression includes all numbers divisible by a, and the second all divisible by b, the latter expression must be included in each of the former; hence, we have,

$a^{m-1} b^n - a^{m-1} b^{n-1}$ terms divisible by a only.

$a^m b^{n-1} - a^{m-1} b^{n-1}$ terms divisible by b only.

$a^{m-1} b^{n-1}$ terms divisible by $a\,b$.

and these together include all those terms of the series 1, 2, 3, 4, &c. $a^m b^n$, that have any divisor common to $a^m b^n$ or N, consequently their sum taken from N will be the number of those that are prime to it, hence that number will

$$= a^m b^n - a^{m-1} b^n - a^n b^{n-1} + a^{m-1} b^{n-1}$$

$$= (a^m - a^{m-1}) b^n - (a^m - a^{m-1}) b^{n-1}$$

$$= (a^m - a^{m-1}) \times (b^n - b^{n-1})$$

$$= \left(a^m \frac{a-1}{a}\right)\left(b^n \frac{b-1}{b}\right) = N \times \frac{a-1}{a} \times \frac{b-1}{b}.$$

And, exactly in the same manner, if N were the product of a greater number of factors, we should still find that the number of integers less than, and prime to, N, would be represented by the formula

$$N \times \frac{a-1}{a} \times \frac{b-1}{b} \times \frac{c-1}{c} \&c.$$

Ex. $100 = 2^2 . 5^2$ and $100 + \frac{2-1}{2} + \frac{5-1}{5} = 40$,

which is the number of integers less than and prime to 100.

(a). If p be the modulus of a form for prime numbers, Art. 156 (a); the number of forms of which that modulus is susceptible will be equal to the number of integers that are less than $\frac{p}{2}$ and prime to it, if $\frac{p}{2} = a^m b^n c^p$ &c. this number will be

$$\frac{p}{2} \times \frac{a-1}{a} \times \frac{b-1}{b} \times \frac{c-1}{c} \&c.$$

Ex. If $p = 20$, $\frac{p}{2} = 10 = 2 \cdot 5$, number of forms

$$= 10 \times \frac{2-1}{2} \times \frac{5-1}{5} = 4,$$

the forms themselves are,

$$20n \pm 1,\ 20n \pm 3,\ 20n \pm 7, \text{ and } 20n \pm 9.$$

171. *To find a perfect number, that is, one which is equal to the sum of its divisors.*

Suppose $N = a^m b$, a perfect number; a and b being its prime factors.

Then, (Art. 168 (a).)

$$\text{sum of its divisors} = \frac{a^{m+1}-1}{a-1} \times \frac{b^1-1}{b-1} = \frac{a^{m+1}-1}{a-1}(b+1).$$

But this includes N itself (Art. 168 (b).) which in the case of a perfect number must be excluded: hence we have,

$$a^m b = \frac{a^{m+1}-1}{a-1}(b+1) - a^m b;$$

or, $2a^{m+1}b - 2a^m b = a^{m+1}b + a^{m+1} - b - 1$;

$$\text{and } \therefore b = \frac{a^{m+1}-1}{a^{m+1} - 2a^m + 1}.$$

In order that b may be an integer, assume

$$a^{m+1} - 2a^m = 0, \text{ or } a = 2; \text{ then, } b = 2^{m+1} - 1;$$

$$\text{and } N = a^m b = 2^m(2^{m+1} - 1);$$

where it must be observed, that since b or $2^{m+1} - 1$ is supposed a prime number, m must be so assumed that this may always be the case.

Ex. If $m = 1$, the number is $(2^2 - 1) \cdot 2 = 6$, which is the least perfect number.

CHAPTER XII.

ON THE INDETERMINATE ANALYSIS.

172. An indeterminate equation is one where two unknown quantities are involved, and there is no other equation expressing a relation between them; the number of solutions will in that case be infinite, as any value whatever being put for one unknown quantity will give a value of the other; but the number of solutions may be restricted by taking only the integral, or only the positive integral, values of the two unknown quantities.

173. The general form of a simple indeterminate equation, when the unknown quantities are integral, is

$$ax \pm by = \pm c,$$

in this equation a and b, are prime to each other, for if they had a common divisor d, the values of x and y being at the same time integral, we should have,

$$\frac{ax \pm by}{d}, \text{ (an integer)} = \pm \frac{c}{d},$$

which must therefore be an integer; hence the equation may be reduced by dividing by the common divisor.

174. *To obtain a solution in whole numbers of the equation*

$$a\,x + b\,y = c.$$

We have, $x = \dfrac{c - b\,y}{a}$, a whole number.

Let $\dfrac{c}{a} = p + \dfrac{c'}{a}$, $\dfrac{b}{a} = q + \dfrac{b'}{a}$;

$$\therefore x = p - q\,y + \frac{c' - b'\,y}{a}, \text{ a whole number;}$$

Whence, taking the fractional part $\dfrac{c' - b'\,y}{a}$, and equating it to a new unknown quantity u, we have

$$u = \frac{c' - b'\,y}{a}, \quad \therefore y = \frac{c' - a\,u}{b'}.$$

Let $\dfrac{c'}{b'} = p' + \dfrac{c''}{b'}$, $\quad \dfrac{a}{b'} = q' + \dfrac{a'}{b'}$;

$$\therefore y = p' - q'\,u + \frac{c'' - a'\,u}{b'}.$$

It is evident that the remainders, being obtained in the same manner as those in finding the greatest common measure, will continually decrease, and by proceeding as above, putting the fractional part $\dfrac{c'' - a'\,u}{b'} = z$, we shall have at last one of the remainders b', a', &c. $= 1$. Suppose $a' = 1$; then

$$\frac{c'' - u}{b'} = z; \quad \therefore u = c'' - b'\,z;$$

$$\therefore y = p' - q'\,(c'' - b'\,z) + z \text{ is known;}$$

$$\text{and } x = p - q\,y + c'' - b'\,z \text{ is also known;}$$

where any value at pleasure may be given to z.

(*a*). If a and c, or b and c, be not prime to each other the solution may be abridged.

Let m be the common measure of a and c.

Then $\dfrac{a\,x}{m} + \dfrac{b\,y}{m} = \dfrac{c}{m}$, also $\dfrac{b\,y}{m}$ must be a whole number,

because by hypothesis all the other terms are whole numbers, therefore y must be of the form $m\,y'$, and by substituting $m\,y'$ for y in the equation $a\,x + b\,y = c$, and dividing by m, we get an equation to solve in lower terms.

175. *The several values of* x *and* y *in the equation* a x + b y = c *are in Arithmetic Progression.*

For, let $x = \alpha$, and $y = \beta$, be one solution of this equation.

Then $a\,\alpha + b\,\beta = c$, and subtracting this from the first equation, we have,

$$a\,(x - \alpha) = -\,b\,(y - \beta), \text{ or, } \frac{a}{b} = -\,\frac{y - \beta}{x - \alpha} = \frac{y - \beta}{\alpha - x};$$

and since a and b are prime to each other, $y - \beta$ must be a multiple (as $\lambda\,a$) of a; Art. 159; and, $\alpha - x$ must be the same multiple (as $\lambda\,b$) of b: that is, $x = \alpha - \lambda\,b$; and $y = \beta + \lambda\,a$; the integral quantity λ being entirely arbitrary. These are general values of x and y, and all others are comprised in these forms; for if $x = \alpha'$; $y = \beta'$ be another solution of the first equation, we have

$$a\,\alpha' + b\,\beta' = c;$$

$$\text{also, } a\,\alpha + b\,\beta = c;$$

$$\therefore\ a\,(\alpha' - \alpha) = -\,b\,(\beta' - \beta)$$

and reasoning as before, $\beta' - \beta$ is a multiple $(\lambda\,a)$ of a;

consequently, $\beta' = \beta + \lambda\,a$; and similarly, $\alpha' = \alpha - \lambda\,b$, which values are comprised in the above forms.

Hence if we make $\lambda = 1, 2, 3,$ &c. successively, the series of values of x and y will form Arithmetic Progressions, of which a, and $-\,b$, are the common ratios.

(*a*). We may hence find the number of *positive* integral solutions which the equation admits of; for since $x = \alpha - \lambda\,b$ and $y = \beta + \lambda\,a$, find the value of λ which makes each of these quantities first negative, the number of values of λ comprised between these two extremes, will give the number of positive integral solutions.

Ex. $5x + 7y = 316$. By proceeding as in Article 172, we find $x = 66 - 7t$, and $t = 10$ first makes this negative;
also, $y = 5t - 2$, and $t = 0$ first makes this negative;
$\therefore$ number of positive integral solutions $= 9$.

176. If an equation were proposed involving three unknown quantities; as

$$ax + by + cz = d, \text{ we have}$$
$$ax + by = d - cz = c', \text{ suppose;}$$

whence we may find values of x and y of the form

$$x = mr + nc', \; y = m'r + n'c',$$

or $x = mr + n(d - cz)$ and $y = m'r + n'(d - cz)$,
where z and r may be taken at pleasure, except in so far as the values of x, y and z are restricted to positive values.

177. Indeterminate equations of the second degree will always produce equations of one of the following forms:

$$(1), y = \frac{a}{b + cx}; \; (2), y = \frac{a + bx}{c + dx}; \; (3), y = \sqrt{a + bx + cx^2}.$$

In the two first of these equations, x is to be determined so that y may be an integer; in the last, so that y may be rational

1st. $y = \frac{a}{b + cx}$;

here $b + cx$ must be a divisor (d) of a; hence

$$b + cx = d \text{ and } x = \frac{d - b}{c},$$

hence we must search among the divisors of a, for one such that when b is subtracted from it the remainder may be divisible by c, the quotient will be the required value of x.

2ndly. $y = \frac{a + bx}{c + dx}$;

if d be a divisor of b, x will be taken out of the numerator, if we divide it by $c + dx$, and this form is then reduced to the

preceding. But if d be not a divisor of b, multiply both sides by d, then

$$dy = \frac{da + dbx}{c + dx}; \text{ or, } dy = b + \frac{ad - bc}{c + dx};$$

hence x is found by making $c + dx$ equal to a divisor of $ad - bc$.

3dly.. $y = \sqrt{a + bx + cx^2}$.

Case 1. Let a be a square number and $= g^2$.

Suppose $\sqrt{g^2 + bx + cx^2} = g + mx$,

then, squaring,

$$g^2 + bx + cx^2 = g^2 + 2gmx + m^2x^2, \text{ or,}$$
$$b + cx = 2gm + m^2x,$$

$$\text{whence, } x = \frac{2gm - b}{c - m^2};$$

$$\text{and, } y = \sqrt{g^2 + bx + cx^2} = \frac{cg - bm + gm^2}{c - m^2}.$$

Here m may be any rational quantity, integral or fractional.

Case 2. Let c be a square number $= g^2$, then putting $\sqrt{a + bx + g^2x^2} = m + gx$ we find

$$a + bx = m^2 + 2gmx, \text{ whence, } x = \frac{m^2 - a}{b - 2mg}.$$

$$\text{and } y = \sqrt{a + bx + g^2x^2} = \frac{bm - gm^2 - ag}{b - 2mg}.$$

Here m, as before, is arbitrary.

Case 3. When neither a nor c is a square number; but the expression $a + bx + cx^2$ can be resolved into two simple factors, $f + gx$; and $f' + g'x$; we have,

$$\sqrt{a + bx + cx^2} = \sqrt{(f + gx)(f' + g'x)} = m(f + gx).$$

Then, $(f + gx)(f' + g'x) = m^2(f + gx)^2$,

$$\text{or, } f' + g'x = m^2(f + gx);$$

$$\text{whence, } x = \frac{m^2f - f'}{g' - m^2g};$$

$$\text{and, } y = \sqrt{(f + gx)(f' + g'x)} = \frac{m(fg - gf')}{g' - m^2g};$$

m being, as before, arbitrary.

Applicable only when $b^2 - 4ac = \square$

Case 4. When $a + bx + cx^2$ is of the form $p^2 + qr$ when p, q, and r, are any quantities which contain no higher power of x than the first.

Let us assume,

$$\sqrt{a + bx + cx^2}\ (= \sqrt{p^2 + qr}) = p + mq;$$

$$\text{then, } p^2 + qr = p^2 + 2mpq + m^2q^2,$$

$$\text{and, } r = 2mp + m^2q;$$

and as this equation involves only the first power of x, we may, by reduction, obtain from it rational values of x and y as in the three foregoing cases.

Ex. To find two square numbers whose sum is a given square, or in other words to find the formula for the sides of a right-angled triangle. (See Euclid I. 45.)

Let a^2 be the given square; x^2, y^2, the numbers required, then, $y = \sqrt{a^2 - x^2}$. This form comes under *Case* 1, so that we have, $g = a$, $b = 0$, $c = -1$, and substituting these values in the formula, and $-n$ for m, we find

$$x = \frac{2na}{n^2 + 1},\ y = \frac{a(n^2 - 1)}{n^2 + 1};$$

hence the numbers required are

$$x^2 = \frac{4a^2n^2}{(n^2 + 1)^2} \text{ and } y^2 = \frac{a^2(n^2 - 1)^2}{(n^2 + 1)},$$

if $a = n^2 + 1$, (n being arbitrary); the values of x^2 and y^2 will both be integers, viz.

$$x^2 = 4n^2; \text{ and } y^2 = (n^2 - 1)^2.$$

(*a*). Similarly we may find two square numbers whose difference is a square.

CHAP. XIII.

ON CONTINUED FRACTIONS.

178. *To convert a fraction* $\frac{M}{N}$, *into a continued one.*

Let y_1 be the nearest integer in $\frac{M}{N}$, and let $\frac{M}{N} = y_1 + \frac{1}{x_1}$, x_1 being the reciprocal of a proper fraction, and therefore an improper one; take y_2 the nearest integer less than x_1, and let $x_1 = y_2 + \frac{1}{x_2}$;

Similarly, suppose, $x_2 = y_3 + \frac{1}{x_3}$ &c. = &c.

Then $\frac{M}{N} = y_1 + \frac{1}{x_1} = y_1 + \frac{1}{y_2 + \frac{1}{x_2}} = y_1 + \frac{1}{y_2 + \frac{1}{y_3 + \frac{1}{x_3}}}$

$$= y_1 + \frac{1}{y_2 + \frac{1}{y_3 + \frac{1}{y_4 + \&c.}}}$$

The fraction (or whole number) resulting from the first term (y_1) of this continued fraction, (neglecting other terms), we shall denote by $\frac{p_1}{q_1}$; that from the two first terms $\left(y_1 + \frac{1}{y_2}\right)$,

by $\frac{p_2}{q_2}$; that from the three first $\left\{y_1 + \frac{1}{y_2 + \frac{1}{y_3}}\right\}$ by $\frac{p_3}{q_3}$; &c.

and that from the n first by $\frac{p_n}{q_n}$;

These will evidently be successive approximations to the value of $\frac{M}{N}$ increasing in accuracy as n increases.

179. *In any continued fraction,* $\frac{p_n}{q_n} = \frac{y_n p_{n-1} + p_{n-2}}{y_n q_{n-1} + q_{n-2}}$.

For, $\frac{p_1}{q_1} = y_1$ where $p_1 = y_1$ and $q_1 = 1$

$$\frac{p_2}{q_2} = y_1 + \frac{1}{y_2} = \frac{y_1 y_2 + 1}{y_2} = \frac{y_2 p_1 + 1}{y_2 q_1}$$

$$\frac{p_3}{q_3} = y_1 + \frac{1}{y_2 + \frac{1}{y_3}} = y_1 + \frac{1}{\frac{y_2 y_3 + 1}{y_3}}$$

$$= \frac{y_1 y_2 y_3 + y_1 + y_3}{y_2 y_3 + 1} = \frac{y_3 (y_1 y_2 + 1) + y_1}{y_3 y_2 + 1}$$

$$= \frac{y_3 p_2 + p_1}{y_3 q_2 + q_1}.$$

Similarly, $\frac{p_4}{q_4} = \frac{y_4 p_3 + p_2}{y_4 q_3 + q_2}$; and generally,

$$\frac{p_n}{q_n} = \frac{y_n p_{n-1} + p_{n-2}}{y_n q_{n-1} + q_{n-2}}.$$

(*a*). If in this last equation we replace y_n by z, where $z =$ the n^{th} together with all the succeeding quotients

$$= \cfrac{1}{y_n + \cfrac{1}{y_{n+1} + \cfrac{1}{y_{n+2} + \text{\&c.}}}}$$

in that case $\frac{p_n}{q_n}$ will become $\frac{M}{N}$; so that we shall thus have the

exact value of the original fraction; that is,

$$\frac{M}{N} = \frac{z\,p_{n-1} + p_{n-2}}{z\,q_{n-1} + q_{n-2}};$$

The quantity z is called *a complete quotient;*

and, $\frac{z\,p_{n-1} + p_{n-2}}{z\,q_{n-1} + q_{n-2}}$ *a complete fraction.*

(*b*). We have $p_n = y_n\,p_{n-1} + p_{n-2}$,

whence, $y_n = \frac{p_n - p_{n-2}}{p_{n-1}}$,

also, $q_n = y_n\,q_{n-1} + q_{n-2}$,

whence, $y_n = \frac{q_n - q_{n-2}}{q_{n-1}}$,

Equate these values of y_n and clear the equation of fractions, then,

$$q_{n-1}\,(p_n - p_{n-2}) = p_{n-1}\,(q_n - q_{n-2}),$$

$$\text{or, } p_n q_{n-1} - q_n p_{n-1} = -\,(p_{n-1}\,q_{n-2} - q_{n-1}\,p_{n-2}),$$

which last, by similar reasoning,

$$= p_{n-2}\,q_{n-3} - q_{n-2}\,p_{n-3}$$

$$= \pm\,\&c.$$

until we come to

$$\pm\,(p_2\,q_1 - q_2\,p_1), \text{ which (Art. 179)}$$

$$= \pm\,\{(y_1\,y_2 + 1)\,1 - y_2\,y_1\} = \pm\,1,$$

the positive sign being used when n is an even number, and the negative when n is odd.

(*c*). By dividing the equation

$p_n\,q_{n-1} - q_n\,p_{n-1} = \pm\,1$, by $q_n\,q_{n-1}$, we get,

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = +\,\frac{1}{q_n\,q_{n-1}}$$

which therefore gives the difference between any two successive approximations.

(*d*). It appears from the equation $p_n q_{n-1} - q_n p_{n-1} = \pm 1$, that any quantity which divides both p_n and q_n must also divide ± 1; or, in other words, the fractions $\frac{p_n}{q_n}$, $\frac{p_{n-1}}{q_{n-1}}$, &c. are all irreducible.

180. *Each successive approximation approaches nearer to the value of* $\frac{M}{N}$ *than the one which precedes it.*

$$\text{For } \frac{M}{N} - \frac{p_{n-1}}{q_{n-1}} = \frac{z\,p_{n-1} + p_{n-2}}{z\,q_{n-1} + q_{n-2}} - \frac{p_{n-1}}{q_{n-1}}; \text{ Art. 177 } (a);$$

$$= \frac{p_{n-2}\,q_{n-1} - p_{n-1}\,q_{n-2}}{q_{n-1}\,(z\,q_{n-1} + q_{n-2})} = \frac{\pm 1}{q^n{}_{-1}\,(z\,q^n{}_{-1} + q_{n-2})}; \text{ Art. 177 } (b),$$

$$\text{Again, } \frac{M}{N} - \frac{p_{n-2}}{q_{n-2}} = \frac{z\,p_{n-1} + p_{n-2}}{z\,q_{n-1} + q_{n-2}} - \frac{p_{n-2}}{q_{n-2}}$$

$$= \frac{z\,(p_{n-1}q_{n-2} - q_{n-1}\,p_{n-2})}{q_{n-2}\,(z\,q_{n-1} + q_{n-2})} = \frac{\pm z}{q_{n-2}\,(z\,q_{n-1} + q_{n-2})}$$

This difference is greater than the former, because z is necessarily greater than 1; and q_{n-2} less than q_{n-1}; hence the approximation $\frac{p_{n-1}}{q_{n-1}}$ is more accurate than $\frac{p_{n-2}}{q_{n-2}}$.

(*a*). Since $\frac{M}{N} - \frac{p_{n-1}}{q_{n-1}} = \frac{\pm 1}{q_{n-1}\,(z\,q_{n-1} + q_{n-2})}$

and that z cannot be less than 1, hence, (changing $n-1$ into n,) the difference between $\frac{M}{N}$ and the n^{th} approximation, $\frac{p_n}{q_n}$, can never be greater than $\frac{\pm 1}{q_n{}^2 + q_n\,q_{n-1}}$; nor *a fortiori* than $\frac{\pm 1}{q_n{}^2}$; or than $\frac{\pm 1}{q_n\,q_{n-1}}$.

(*b*). Since, $\frac{M}{N} - \frac{p_n}{q_n} = \frac{\pm 1}{q_n^2 + q_n q_{n-1}}$, nearly, if the positive sign be taken, (that is, if n be even, Art. 177 (*b*)) it appears that the approximation $\frac{p_n}{q_n}$ is less than the true value of $\frac{M}{N}$; and if the negative sign be taken, it is greater; or the *even* approximations are too great, and the *odd* ones too little.

(*c*). Since in the approximation $\frac{p_n}{q_n}$ a part of the whole denominator of $\frac{M}{N}$ has been omitted, it is evident that the fraction $\frac{M}{N}$ is in higher terms than $\frac{p_n}{q_n}$; also $\frac{p_n}{q_n}$ is an irreducible fraction; (Art. 177 (*d*)); hence no other fraction $\frac{M'}{N'}$ in lower terms than $\frac{p_n}{q_n}$ is a nearer approximation to $\frac{M}{N}$ than is $\frac{p_n}{q_n}$.

181. The theory of continued fractions is of considerable utility in furnishing an expeditious solution of an indeterminate equation;

Let, $a x + b y = c$, be the equation, and convert the fraction $\frac{a}{b}$ into a continued one; $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, &c. being the successive convergents; suppose we have obtained the complete value of $\frac{a}{b}$ when we have taken the n^{th} convergent, that is, let $\frac{p_n}{q_n} = \frac{a}{b}$; then, Art. 177 (*b*);

$$p_n q_{n-1} - q_n p_{n-1} \; (= a q_{n-1} - b p_{n-1}) = \pm 1$$

$$\text{hence, } a c q_{n-1} - b c p_{n-1} = \pm c$$

comparing this with the original equation, we have

$$c\,q_{n-1} = x \text{ and } -c\,p_{n-1} = y,$$

which are, therefore, solutions of the equation.

Ex. $105\,x - 43\,y = 17$, by following the method in Art. 178. we should find the $\overline{n-1}^{\text{th}}$ or 5^{th} convergent, towards the fraction $\frac{105}{43}$ to be $\frac{22}{9}$, hence $105 \times 9 - 43 \times 22 = -1$, (where the $-$ sign is used because the continued fraction has 5 terms, Art. 179 (*b*); hence, multiplying this equation by -17, and comparing it with the given one, we have

$$x = -19 + 17 \text{ and } y = -22 \times 17,$$

whence, (Art. 173); $x = -153 + 43\,t$; and $y = -374 + 105\,t$. which comprize all the values of x and y.

182. The most accurate observations have determined the length of the solar tropical year to be 365,242264 days.

This not being an exact number of days, requires the *intercalation* of a day at stated intervals of time, in order to prevent the commencement of the civil year from happening at different seasons; to determine how this intercalation must be made, we must reduce the part $\frac{242264}{1000000}$ to a continued fraction, its successive convergents are $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{39}{161}$, $\frac{86}{355}$, &c. Julius Cæsar proceeded upon the principle of the first of these convergents and intercalated a day every 4 years; the ancient Persians adapted a complicated system of intercalation from the third convergent $\frac{8}{33}$; (See *Francœur's Complete Course*, Art. 564). The Gregorian correction, which is the one used at present, intercalates one day every 4 years, omitting the intercalation every 100 years, three times out of four; in this way 97 days are intercalated every 400 years. This intercalation is not quite exact, since the fraction $\frac{97}{400}$ is not found amongst the

above convergents; but the error is too small to produce any sensible inconvenience until after the lapse of many centuries, for the fraction $\frac{97}{400}$ differs little from the second convergent, $\frac{7}{29}$, and the error introduced by taking this last instead of the whole fraction, would be less than $\frac{1}{29^2}$, Art. 178 (a); or less than the $\frac{1}{841^{\text{th}}}$ part of a day.

183. If we have the equation $u = a^x$ we may find an approximate value of x by means of continued fractions; for, let the highest exact power of a contained in u be a^n, so that a^n is $< u$; and $a^{n+1} > u$; hence the value of x lies between n and $n + 1$;

$$\text{Let } x = n + \frac{1}{x'}, \text{ hence,}$$

$$a^{n+\frac{1}{x'}} = u, \text{ or, } a^n . a^{\frac{1}{x'}} = u.$$

$$\therefore \left(\frac{u}{a^n}\right)^{x'} = a.$$

In the same manner suppose in this equation, that x' lies between n' and $n' + 1$; let $x' = n' + \frac{1}{x''}$; and proceeding thus we shall have at last

$$x = n + \cfrac{1}{n' + \cfrac{1}{n'' + \cfrac{1}{n''' + \&c.}}}$$

(a). It appears from Art. 178 (a) that $n + \frac{1}{n'}$, differs from x, by a quantity less than $\frac{1}{n(n+n')}$; and that the third approximation differs from x by a quantity less than

$$\frac{1}{(n'n'' + n' + 1)(n'n'' + 1)}.$$

184. *To extract the root of a number in the form of a continued fraction.*

Suppose 19 the number, then since 4 is the greatest integer root contained in it, take

$$\sqrt{19} = 4 + \frac{1}{y}, \text{ whence } y = \frac{1}{\sqrt{19} - 4} = \frac{\sqrt{19} + 4}{3};$$

Again, since 2 is the greatest integer in $\frac{\sqrt{19} + 4}{3}$, take

$$\frac{\sqrt{19}+4}{3} = 2 + \frac{1}{y'}, \text{ whence } y' = \frac{3}{\sqrt{19}-2} = \frac{3(\sqrt{19}+2)}{15} = \frac{\sqrt{19}+2}{5};$$

similarly, $\frac{\sqrt{19} + 2}{5} = 1 + \frac{1}{y''}$, and $y'' = \frac{5}{\sqrt{19} - 3} = \frac{\sqrt{19}+3}{2}$;

also, $\frac{\sqrt{19} + 3}{2} = 3 + \frac{1}{y'''}$, and $y''' = \frac{2}{\sqrt{19}-3} = \frac{\sqrt{19} + 3}{5}$;

By pursuing the same process we shall at length, after three more operations, reproduce the original fraction $\frac{1}{\sqrt{19} - 4}$; after which the convergents will of course re-appear in the same order as before, and by substituting for y, y', y'', &c. their values, we shall have,

$$\sqrt{19} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{8 + \&c.}}}}}}$$

(*a*). By continuing the above process far enough, we shall find that in all cases the resulting convergents will be reproduced in a certain periodic order; for a proof of this, and an explanation of the theory upon which the above is founded, see *Francœur's Complete Course*, vol. 2, p. 123.

(*b*). We may, by observation and induction, deduce from the process above employed in extracting the root of 19, a law for the formation of the successive quotients;

$$\text{for if } \frac{\sqrt{19} + m}{n} = n + \&c. \text{ and } \frac{\sqrt{19} + m'}{n'} = u' + \&c.$$

represent any two consecutive fractions, u and u' being their respective quotients, it is easily seen that,

$$m' = n\,u - m; \text{ and } n' = \frac{19 - m'^2}{n};$$

so that each successive value of m', n', and u', may be thus deduced from those in the preceding operation.

CHAP. XIV.

ON THE RESOLUTION OF COMPOUND FRACTIONS INTO THEIR COMPONENT SIMPLE FRACTIONS.

185. *If* $a_0 + a_1 x + a_2 x^2 + \&c. + a_n x^n = b_0 + b_1 x + b_2 x^2 + \&c. + b_n x^n$ *for all values of* x, *then will* $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$ &c. and $a_n = b_n$.

For, let $x = o$; then we have, $a_0 = b_0$; an equality which is not rendered such, by giving a particular value to x, but which exists independently of that particular value; hence we shall still have, for all values of x,

$$a_1 x + a_2 x^2 + \&c. + a_n x = b_1 x + b_2 x^2 + \&c. + b_n x^n;$$

whence, dividing by x, and supposing x again $= o$, we shall have, $a_1 = b_1$; and similarly, $a_2 = b_2$; and generally, $a_n = b_n$.

186. *To decompose the compound rational fraction* $\frac{N}{D}$, *into its simple component fractions.*

If the numerator be not of lower dimensions, as respects the powers of x, than the denominator, it must first be made so by actual division;—

Let $D = d\, d'\, d''$ &c., where d, d', d'', &c. are polynomial factors prime to each other, of the dimensions δ, δ', δ'', &c. respectively, and assume

$$\frac{N}{D} = \frac{A x^{\delta-1} + B x^{\delta-2} + \&c. + L}{d} + \frac{A' x^{\delta'-1} + B' x^{\delta'-2} + \&c. + L'}{d'} + \&c.$$

reduce the fractions on the second side to a common denominator, the numerators will then evidently be each of the dimensions $\delta + \delta' +$ &c. $- 1$; and the common denominator will be $d\,d'\,d''$ &c. $= D$; hence by equating co-efficients of the same powers of x on each side, we shall have, $\delta + \delta' + \delta''$ &c. equations to determine the unknown co-efficients A, A', &c. B, B', &c. the number of which will evidently be $\delta + \delta' + \delta'' +$ &c. since the numerators contain each δ, δ', δ'', &c. terms respectively.

(a). The denominator D, must first be equated to 0, in order to determine its prime factors.

187. There are several cases of the above, which we shall examine separately,

1st. If D be of the form $(x - a)\,(x - b)\,(x - c)$, we must assume,

$$\frac{N}{D} = \frac{A}{x - a} + \frac{B}{x - b} + \frac{C}{x - c} + \text{&c.}$$

2d. When D contains imaginary binomial factors; it is generally preferable to decompose it into real trinomials of the form $x^2 + p\,x + q$; thus if $D = (x - a)\,(x^2 + b^2)$, we should assume,

$$\frac{N}{D} = \frac{A}{x - a} + \frac{B\,x + L}{x^2 + b^2}.$$

3dly. When D contains factors of the form $(x - a)^n$; we might deduce such a simple component fraction as

$$\frac{A\,x^{n-1} + B\,x^{n-2} + \text{&c.}}{(x - a)^n},$$

but this being again decomposable, we may at once assume, instead of it, the equivalent fractions,

$$\frac{A}{(x - a)^n} + \frac{B}{(x - a)^{n-1}} + \frac{C}{(x - a)^{n-2}} + \text{&c.} + \frac{L}{x - a},$$

the other factors of D (if there be any others,) being treated as before.

4thly. If D contain equal factors of an imaginary form; we must assume, as in the second case, fractions of the form,

$$\frac{Ax+B}{(x^2+px+q)^n}+\frac{Cx+D}{(x^2+px+q)^{n-1}}+\text{&c.}+\frac{Kx+L}{x^2+px+q}$$

and proceed as before.

188. If some of the prime factors of the denominator be known, the fraction may be resolved into others, by the following method, which will be found less laborious than those given above.

Suppose one factor $x-a$ of the denominator known, and let $\frac{N}{D}=\frac{N}{(x-a)D'}$

assume, $\frac{N}{(x-a)D'}=\frac{A}{x-a}+\frac{P}{D'}=\frac{AD'+(x-a)P}{(x-a)D'}$,

and the object is, to determine A, without knowing the polynomial P; we have then,

$$(x-a)P=N-AD';\quad\therefore\ P=\frac{N-AD'}{x-a};$$

which last is not fractional, because P is supposed a rational quantity involving no negative powers of x; hence $N-AD'$ must be divisible by $x-a$ exactly: Let now $x-a=0$, or $x=a$, then in order that P may not, upon this supposition, become infinite, $N-AD'$ must $=0$, likewise; Art. 189.

$$\therefore A=\frac{N_a}{D'_a};$$

N_a and D'_a being the values of N and D' when $x=a$; and similarly the numerators of the other component fractions may be determined.

189. If D contain factors of the form $(x-a)^n$ we must assume,

$$\frac{N}{D} = \frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} + \&c. + \frac{L}{x-a} + \frac{P}{D'}$$

$$= \frac{AD' + BD'(x-a) + \&c. + LD'(x-a)^{n-1} + P(x-a)^n}{D}$$

$$\therefore P = \frac{N - AD' - BD'(x-a) - \&c. - LD'(x-a)^{n-1}}{(x-a)^n}$$

which must be exactly divisible by $x-a$; and as before, we must have when $x=a$, $N-AD'=0$, since every other term of the numerator involves $x-a$; therefore

$$A = \frac{N_a}{D'_a}.$$

Again make, $\frac{N-AD'}{x-a} = {}^1N$, and therefore

$$P = \frac{{}^1N - BD' - \&c. - LD'(x-a)^{n-2}}{(x-a)^{n-1}}$$

whence, as before, $B = \frac{{}^1N_a}{D'_a}$, and similarly if

$$\frac{{}^1N - BD'}{x-a} = {}^2N; \text{ and, } \frac{{}^2N - CD'}{x-a} = {}^3N, \&c.$$

we shall have, $C = \frac{{}^2N_a}{D'_a}$, &c. until the quantities A, B, C, &c. are all determined.

190. When a fraction is proposed whose denominator contains both equal and unequal factors, it is generally best to treat the unequal factors by the first case of Art. 185; and then, having subtracted the component fractions corresponding to them from the original fraction, to treat the remaining fraction by the method explained in Art. 187.

Ex. Let the proposed fraction be

$$\frac{N}{D} = \frac{x^3 - 6x^2 + 4x - 1}{(x-2)(x-3)(x+1)^2}$$

and assume $\frac{N}{D} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{P}{D'}$

Then, first,

$N = x^3 - 6x^2 + 4x - 1$; $\therefore N_a = 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1 = -9$

$D' = (x-3)(x+1)^2$; $\therefore D'_a = (2-1)(2+1)^2 = -9$

$$\text{Hence, } A = \frac{N_a}{D'_a} = \frac{-9}{-9} = 1;$$

again, to determine B, we have,

$N = x^3 - 6x^2 + 4x - 1$ $\therefore N_b = 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1 = -16$

$D' = (x-2)(x+1)^2$ $\therefore D'_b = 1 \cdot 4^2 = 16$

$$\text{Hence } B = \frac{N_b}{D'_b} = \frac{-16}{16} = -1;$$

and the two fractions, corresponding to the unequal factors, are

$$\frac{1}{x-2} - \frac{1}{x-3};$$

subtract this from the original fraction, and the result is $\frac{x}{(x+1)^2}$, which remains to be decomposed; we have then,

$$\frac{x}{(x+1)^2} = \frac{A}{(x+1)^2} + \frac{B}{x+1};$$

$\therefore x = A + Bx + B$, whence, (Art. 183.)

$\left.\begin{matrix} A + B = 0 \\ B = 1 \end{matrix}\right\}$ and therefore $A = -1$; and finally,

$$\frac{x^3 - 6x^2 + 4x - 1}{(x-2)(x-3)(x+1)^2} = \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{(x+1)^2} + \frac{1}{x+1}.$$

191. It sometimes happens, when a particular value (a) is given to the indeterminate quantity (x) which enters into a fraction, that the fraction itself will then assume the form of $\frac{0}{0}$; this

arises from some modification of $(x - a)$ entering into the numerator and denominator; the real value of the fraction may be determined in such a case, by putting $a + h$ for x, then developing two or three terms of any binomials that may remain, and afterwards making $h = 0$.

Ex. To find the value of the fraction

$$\frac{(x^2 - a^2)^{\frac{3}{2}} + x - a}{(1 + x - a)^3 - 1}, \text{ when } x = a,$$

putting $a + h$ for x, and developing, it becomes

$$\frac{h + (2\,a\,h)^{\frac{3}{2}} \ldots\ldots}{3\,h + 3\,h^2 \ldots\ldots};$$

dividing numerator and denominator by h, and making $h = 0$, we have $\frac{1}{3}$ for the real value of the fraction when $x = a$.

(a). When $x = a$ causes the product PQ to assume the form $O \times \infty$; since $\frac{1}{O} = \infty$, we shall have $\frac{P}{Q} = \frac{O}{O}$, which comes under the last form.

(b). Similarly, when $\frac{P}{Q}$ becomes $\frac{\infty}{\infty}$, then $\frac{Q}{P} = \frac{O}{O}$.

(c). And if an expression assume the form of $\infty - \infty$, this is equivalent to $\frac{1}{P} - \frac{1}{Q}$, where P and Q are each $= O$, or to $\frac{Q - P}{PQ}$, which as before takes the form of $\frac{O}{O}$.

CHAPTER XV.

ON LOGARITHMS.

DEF. 1. IF $u = a^x$, then x is called *the logarithm* of u to *the base* a, denoted thus (log u).

DEF. 2. *A system* of logarithms is a series of values of x corresponding to different values of u, the *same* base a being preserved.

192. *In any system of logarithms*

$$\log\left(\frac{u'\,u''\,u'''\,\ldots\ldots}{u_{,}\,u_{,,}\,u_{,,,}\,\ldots\ldots}\right) = \log u' + \log u'' + \log u''' \ldots\ldots$$

$$- (\log u_{,} + \log u_{,,} + \log u_{,,,} \ldots\ldots$$

For let $u' = a^{x'}$; $u'' = a^{x''}$; $u''' = a^{x'''}$ &c.

Also, let $u_{,} = a^{x_{,}}$; $u_{,,} = a^{x_{,,}}$; $u_{,,,} = a^{x_{,,,}}$, &c.

Then $u_{,}\,u_{,,}\,u_{,,,} \ldots\ldots = a^{x_{,}+x_{,,}+x_{,,,}} \ldots\ldots$

and, $u'\,u''\,u''' \ldots\ldots = a^{x'+x''+x'''} \ldots\ldots$

$$\therefore \frac{u'\,u''\,u'''\,\ldots\ldots}{u_{,}\,u_{,,}\,u_{,,,}\,\ldots\ldots} = a^{x'+x''+x'''} \times a^{-(x_{,}+x_{,,}+x_{,,,})}$$

whence according to the definition of logarithms,

$$x' + x'' + x''' \ldots - (x_{,} + x_{,,} + x_{,,,} \ldots.) = \log\left(\frac{u'\,u''\,u'''\ldots.}{u_{,}\,u_{,,}\,u_{,,,}\ldots.}\right).$$

$$\text{or } \log u' + \log u'' + \log u''' \ldots$$
$$-(\log u_{,} + \log u_{,,} + \log u_{,,,} \ldots) = \log \left(\frac{u' u'' u''' \ldots .}{u_{,} u_{,,} u_{,,,} \ldots .} \right).$$

(*a*). Hence also $\frac{m}{n} \log u = \log u^{\frac{m}{n}}$.

$$\text{For let } u = a^x; \ \therefore \ u^{\frac{m}{n}} = a^{\frac{m}{n}x};$$

$$\therefore \ \frac{m}{n} x = \log u^{\frac{m}{n}}; \text{ or } \frac{m}{n} \log u = \log u^{\frac{m}{n}}.$$

(*b*). Hence if the arithmetical values of the logarithms of numbers be calculated and collected into tables, the multiplication and division of numbers may be performed by the addition and subtraction of their logarithms as given by the tables; and the involution or evolution of numbers by multiplying or dividing their logarithms by the indices of the powers or roots required.

193. *To expand* a^x *in a series proceeding by powers of* x.

$$\text{Let } u = a^x = 1 + k x + k_2 x^2 + k_3 x^3 + \&c.$$

where 1 is put for the first term of the expansion, because when $x = 0$, $a^x = 1$;

$$\text{let } a = 1 + b; \text{ then,}$$

$$u = (1 + b)^x = (1 + b)^{\frac{n x}{n}}$$

n being any arbitrary quantity, whose value is assignable at pleasure.

$$\text{Now, } (1 + b)^{\frac{n x}{n}} = \left\{ (1 + b)^n \right\}^{\frac{x}{n}},$$

$$\text{or } (1 + b)^x = \left\{ 1 + n (k + k_2 n + k_3 n^2 + \&c.) \right\}^{\frac{x}{n}}$$

$$= \left\{ 1 + n (k + P n) \right\}^{\frac{x}{n}} \text{ by putting } P \text{ for } k_2 n + k_3 n^2 + \&c.$$

(which by expanding by the binomial theorem)

$$= 1 + \frac{x}{n} n (k + P n) + \frac{\frac{x}{n}\left(\frac{x}{n} - 1\right)}{1 \cdot 2} n^2 (k + P n)^2 + \&c.$$
$$+ \frac{\frac{x}{n}\left(\frac{x}{n} - 1\right) \ldots \ldots \left(\frac{x}{n} - m - 1\right)}{1 \cdot 2 \cdot 3 \ldots \ldots m} n^m (k + P n)^m$$

$$= 1 + x(k + Pn) + \frac{x(x-n)}{1.2}(k + Pn)^2 + \&c.$$

$$+ \frac{x(x-n)\ldots.(x-m^n+n)}{1.2.3\ldots.m} \cdot (k + Pn)^m \ldots.;$$

now n being an arbitrary quantity cannot enter into the expansion of a^x, we may therefore make $n = 0$, in which case we have

$$a^x \text{ or } (1 + b)^x = 1 + kx + \frac{k^2}{1.2}x^2 + \frac{k^3}{1.2.3}x^3 + \&c.$$

$$+ \frac{k^m x^m}{1.2.3\ldots.m}\ldots.$$

(see Lagrange's "*Calcul des Fonctions*," whence the principle of the above method is taken).

(a). To find the value of k in this expansion, we observe that it is the co-efficient of x in the series for $(1 + b)^x$ which is

$$1 + xb + \frac{x(x-1)}{1.2}b^2 + \frac{x(x-1)(x-2)}{1.2.3}b^3 + \&c.$$

now the multiplication of a number of factors such as $x(x-1)(x-2)\ldots\ldots(x-\overline{n-1})$ together, will produce an expression of the form $x^{n-1} - px^{n-2} + \&c. \ldots \pm Px$, the last co-efficient P being the product

$$(-1)(-2)(-3)\ldots.(-\overline{n-1}) = \pm 1.2.3.4\ldots.\overline{n-1}$$

the sign being + or − according as there is an even or odd number of factors composing the product; hence, we have for the co-efficient of the n^{th} term

$$\pm \frac{1.2.3\ldots.\overline{n-1}}{1.2.3\ldots\ldots n}b^n = \pm \frac{b^n}{n}$$

$$\therefore k = b - \frac{b^2}{2} + \frac{b^3}{3} - \&c. \ldots\ldots \pm \frac{b^n}{n}\ldots\ldots\ldots .$$

$$= (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} \ldots\ldots \pm \frac{(a-1)^n}{n}\ldots.$$

since $a = 1 + b$.

194. *To find a series for the logarithm in terms of its corresponding number.*

This is to find a series for x in terms of u from the equation $u = a^x$; let $u = 1 + y$, and as before $a = 1 + b$.

$$\text{Then } 1 + y = (1 + b)^x;$$

$$\therefore (1 + y)^n = (1 + b)^{nx},$$

n being, as before, entirely arbitrary; hence, expanding by the binomial theorem, we have,

$$1 + ny + \frac{n(n-1)}{1 \cdot 2} y^2 + \&c. = 1 + nxb + \frac{nx(nx-1)}{1 \cdot 2} b^2 + \&c.$$

or, striking out 1 on each side and dividing by n,

$$y + \frac{n-1}{1 \cdot 2} y^2 + \frac{(n-1)(n-2)}{1 \cdot 2 \cdot 3} y^3 + \&c.$$

$$= xb + \frac{x(nx-1)}{1 \cdot 2} b^2 + \frac{x(nx-1)(nx-2)}{1 \cdot 2 \cdot 3} b^3 + \&c.$$

$$\text{Let } n = 0;$$

$$\therefore y - \frac{y^2}{2} + \frac{y^3}{3} - \&c. = xb - \frac{xb^2}{2} + \frac{xb^3}{3} - \&c.$$

$$= x\left(b - \frac{b^2}{2} + \frac{b^3}{3} - \&c.\right)$$

$$= kx; \text{ by Art. 191 } (a).$$

$$\text{hence } x = \text{Log } (1 + y) = \frac{1}{k}\left\{y - \frac{y^2}{2} + \frac{y^3}{3} - \&c.\right\}.$$

or, in terms of u, which $= 1 + y$,

$$x = \text{Log } u = \frac{1}{k}\left\{(u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \&c.\right\}$$

(a). We may find such a value of the base a as will render the quantity $\frac{1}{k}$ unity.

Let e be this value, then

$$e^x = 1 + kx + \frac{k^2}{1 \cdot 2} x^2 + \&c.$$

Let $k = 1$, and since this expansion is true for all values of x we may suppose $x = 1$, likewise;

$$\therefore e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c. = 2{,}7182818,$$

as this is not an exact number, it will be convenient to denote it by e, its incommensurability with any other number may be thus proved.

$$\text{We have } e - 2 = \frac{1}{1.2} + \frac{1}{1.2.3} + \&c.$$

this series is evidently less than the series

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \&c.$$

which last is a geometric series whose sum is 1; hence e is greater than 2, and less than 3; secondly, the value of e cannot be expressed by any finite fraction, as $\frac{m}{n}$; for then we should have,

$$\frac{m}{n} - 2 = \frac{1}{1.2} + \frac{1}{1.2.3} + \&c. + \frac{1}{1.2\ldots n} + \frac{1}{1.2\ldots n.n+1} + \&c.$$

multiply both sides by $1.2.3\ldots.n$;

$$\therefore m.1.2.3\ldots.\overline{n-1} - 1.2^2.3.4\ldots.n$$

$$= 3.4\ldots.n + 4.5\ldots.n + \&c. + 1 + \frac{1}{n+1}$$

$$+ \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \&c.$$

as the first side is a whole number, the second must be so likewise; but the series,

$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \&c.$$

is evidently less than the series

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \&c.$$

which last series $= \frac{1}{n}$, therefore the second side cannot be a whole number, which is absurd.

(*b*). The system of logarithms which has e for its base is called the *Napierien* or *Hyperbolic* system; this logarithm of x is generally written $\log x$; any other logarithm of that quantity being written Log x.

(*c*). Hence in the Napierien system, where $u = e^x$, we have $x \log e = \log u$ and if u be made $= e$ we have $x \log e = \log e$ or x, which (in that case) is the Napierien logarithm of $e = 1$.

(*d*). Since $a^x = 1 + k x + \frac{k}{1.2} x^2 +$ &c.

let $k x = 1$; $\therefore x = \frac{1}{k}$;

hence, $a^{\frac{1}{k}} = 1 + 1 + \frac{1}{1.2} +$ &c. $= e$ by (*a*)

$\therefore e^k = a$; and, k Log e = Log a;

$\therefore k = \frac{\text{Log } a}{\text{Log } e}$ (the base being of any value)

$= \log a$ (when the base $= e$; since then, $\log e = 1$, by (*c*);)

195. *To find a converging series for a logarithm in terms of its corresponding number.*

(By Art. 194) Log $(1 + y) = \frac{1}{k}\left\{y - \frac{y^2}{2} + \frac{y^3}{3} - \text{&c.}\right\}$

for y write $-y$;

$$\therefore \text{Log } (1 - y) = \frac{1}{k}\left\{-y - \frac{y^2}{2} - \frac{y^3}{3} - \text{&c.}\right\}$$

$$\therefore \text{Log}(1+y) - \text{Log}(1-y) = \text{Log}\frac{1+y}{1-y} = \frac{2}{k}\left\{y + \frac{y^3}{3} + \frac{y^5}{5} + \text{&c.}\right\}$$

assume, $\frac{1+y}{1-y} = \frac{b}{c}$,

from which equation, $y = \frac{b-c}{b+c}$;

$$\therefore \log b - \log c = \frac{2}{k}\left\{\frac{b-c}{b+c} + \frac{1}{3}\left(\frac{b-c}{b+c}\right)^3 + \frac{1}{5}\left(\frac{b-c}{b+c}\right)^5 + \text{&c.}\right\}$$

a series by which $\log b$ may be calculated from the logarithm of another quantity c; when the difference between b and c is small,

this series is very convergent, if they be consecutive numbers as n and $n+1$, we have,

$$\text{Log}\,(n+1) = \text{Log}\,n + \frac{2}{k}\left\{\frac{1}{2n+1} + \frac{1}{3\,(2n+1)^3} + \&c.\right\}$$

196. We may obtain other series for the computation of a logarithm in terms of its number;

since (Art. 194) $\text{Log}\,u = \frac{1}{k}\left\{(u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \&c.\right\}$

for u write $u^{\frac{1}{m}}$; then,

$$\text{Log}\,u^{\frac{1}{m}} = \frac{1}{m}\text{Log}\,u = \frac{1}{k}\left\{(u^{\frac{1}{m}} - 1) - \frac{(u^{\frac{1}{m}} - 1)^2}{2} + \frac{(u^{\frac{1}{m}} - 1)^3}{3} - \&c.\right\}$$

$$\text{hence, Log}\,u = \frac{m}{k}\left\{(u^{\frac{1}{m}} - 1) - \left(\frac{u^{\frac{1}{m}} - 1}{2}\right)^2 + \&c.\right\}.$$

In this series, if u be > 1 we may always assume m so large that $u^{\frac{1}{m}} - 1$ may be a very small quantity; and in that case the series will rapidly converge; the arithmetical operations which it involves are however too laborious for practice*.

---

* The following is the method actually employed in the calculation of logarithms of high numbers,

Since by the Calculus of Finite Differences

$$u_{n+x} = u_n + x\Delta u_n + \frac{x\,(x-1)}{1\,.\,2}\Delta^2 u_n + \&c.$$

Let $u_n = \log n$, a quantity previously calculated

$u_{n+x} = \log\,(n+x)$ ..... to be found

here n is supposed a large number and x small in comparison with it; and putting $\Delta = \log\,(n+1) - \log n$; $\Delta' = \log\,(n+2) - \log\,(n+1)$

then $\Delta^2 = \Delta - \Delta'$ and we have,

$\log\,(n+x) = \log n + x\Delta - \frac{x}{2}(\Delta - \Delta')$ nearly, by neglecting terms multiplied by x^2, x^3, &c. as being small compared with the others, and neglecting also, for the same reason, the terms involving the third, fourth, &c. differences. [Hence

(a). Many other series may also be obtained by assuming different values for a and b in Art. 195. See *Peacock's Algebra*, p. 561.

197. The logarithms of numbers which are not prime, may be deduced from those of their component factors.

Thus let $N = a^p b^q c^r \ldots\ldots$

Then $\log N = p \log a + q \log b + c \log r \ldots\ldots$ &c.

198. Having calculated a table of logarithms for a given base a they may be changed to a system whose base is b, thus,

Let $u = b^x$, then by definition, x is the logarithm of u to the base b.

Take the logarithm of this equation in the known system, where the base $= a$;

$$\therefore \text{Log } u = x \text{ Log } b$$

$$\therefore x \text{ which} = \text{Log } u \text{ (to base } b) = \frac{\text{Log } u \text{ (to base } a)}{\text{Log } b \text{ (to base } a}.$$

Hence to transform a system from base a to base b, we have merely to multiply every logarithm in that system, by the reciprocal of the logarithm of the new base b, that logarithm being taken in the *given* system whose base is a.

(a) If the given system be the Napierien, we must, to transform it to base b, multiply each logarithm by $\frac{1}{\log b}$

$$= \frac{1}{b - 1 + \frac{(b-1)^2}{2} + \frac{(b-1)^3}{3} + \text{ \&c.}} \text{ Art. 194.}$$

$$= \frac{1}{k} \text{ Art. 193 } (a).$$

Hence if the logarithms of any three consecutive numbers, n, $n + 1$, $n + 2$, be calculated, we may from this formula find that of $n + x$; but x must not be taken too large compared with n as in that case the terms neglected would produce a sensible influence on the result.

The quantity $\frac{1}{\log b}$ or $\frac{1}{k}$ is called *the modulus* of the system of logarithms which has b for its base; and since in the Napierien system $k = 1$, (Art. 194. *b*.) we have $\log e = 1$, as has been shewn before, Art. 194 (*c*).

199. The numbers selected for the base of the common system of logarithms is 10, let then

$$u = 10^x; \text{ therefore } 10^n u = 10^{x+n}; \text{ and } 10^{-n} u = 10^{x-n}$$

hence denoting by t log, the tabular logarithm, where 10 is the base, we have,

$$t \log 10^n u = x + n = t \log u + n$$

$$\text{and } t \log \frac{u}{10^n} = \ldots\ldots t \log u - n$$

hence in this system, all numbers in decuple proportion (whether whole numbers or decimals) have the same decimal part of their logarithms, the only difference being in the whole number *or characteristic* prefixed to that decimal part

$$\text{Thus if } x = t \log 1132 = 3{,}05384$$

$$\text{then, } x - 2 = t \log 11{,}32 = 1{,}05384$$

$$\text{and } x - 6 = t \log {,}001132 = \bar{3}{,}05384$$

where the sign — is placed over the characteristic, to shew that it must be subtracted from the decimal part of the logarithm, this is done in order that the decimal part of the logarithm may remain the same, for all numbers (whether whole or decimal) expressed by the same digits.

200. It appears from the equation $u = 10^x$, that all numbers between 1 and .1, or $\frac{1}{10}$, have — 1 for their characteristic, all between, 1 and .01; — 2 &c. all between 1 and 10 have 1; between 10 and 100,2; &c. &c. for this reason the tables give only the decimal part or "*mantissa*" of the logarithms, as the characteristic is immediately known from the number of integral places.

Ex. 1. *Given the logarithm* 2,9940398 *to find the number corresponding;*

The decimal part of this logarithm belongs to the digits 98637;

The characteristic 2 shews that the number must lie between 100 and 1000; hence, pointing the digits 98637 so as to fulfil this condition, we have, 2,9940398 = log 986,37.

Ex. 2. *To find the logarithm of,* 00053399.

$1 - \log 53399$ has ,7275331 for its decimal part;

$\therefore \log ,0005399 = 4,7275331.$

201. The advantages of selecting 10 for the base of a system of logarithms intended to facilitate actual calculation will hence be apparent; the Napierien logarithms are however always used in analysis, on account of their greater *analytical* simplicity, owing to their modulus $\left(\frac{1}{k}\right)$ taking the form of unity.

202. Tables of logarithms are calculated for every number up to 10,000; but it may happen that the logarithm of a quantity (as for instance a mixed whole number and decimal) may be required, when there is no such exact quantity in the tables. In that case, suppose N the quantity whose logarithm is required; and let $N = n + x$, where n is the number nearest to it put down in the tables, and x small with respect to n, then

$$t \log N = t \log (n + x) = t \log \left\{ n\left(1 + \frac{x}{n}\right)\right\}$$

$$= t \log n + \frac{x}{k\,n} \text{ nearly, Art. 194;}$$

neglecting $\frac{x^2}{n^2} \cdot \frac{x^3}{n^3}$ &c. as being small compared with the terms preserved.

Now if we make Δ = difference between the logarithm of n, and the logarithm of its consecutive number $n + 1$; we shall have,

$$\Delta = t \log (n + 1) - t \log n = t \log \frac{n+1}{n} = t \log \left(1 + \frac{1}{n}\right)$$

$$= \frac{1}{k n} \text{ nearly};$$

$\therefore$ substituting Δ for $\frac{1}{k n}$ we have $t \log N = t \log n + x \Delta$,

which (since $\Delta = t \log (n + 1) - t \log n$, may be found from the tables,) will give $t \log N$ in terms of known quantities.

(*a*) If the converse be required, viz. to find from a *given* logarithm ($t \log N$) the number N corresponding, such given logarithm not being exactly in the tables, we have

$$x = \frac{t \log N - t \log n}{\Delta}$$

$$\therefore N = n + x = n + \frac{t \log N - t \log n}{\Delta}$$

where $t \log n$ (and $\therefore n$) is known, as being the nearest inferior logarithm in the tables to the given one, $t \log N$.

(*b*) We may hence find the logarithm of a number consisting of r digits, from a table of logarithms constructed for *five* digits only:

For, suppose, n = the number consisting of the first five digits;

x = ... remaining $r - 5$ digits;

N = the number whose logarithm is required;

Then $N = 10^{r-5} n + x$; and as before,

$$\text{and } t \log N = t \log \left\{ 10^{r-5} n \left(1 + \frac{x}{10^{r-5} n}\right) \right\}$$

and substituting Δ for $\frac{1}{k n}$,

$$t \log N = t \log (10^{r-5} n) + \frac{x \Delta}{10^{r-5}}, \text{ very nearly;}$$

$$= r - 5 + t \log n + \frac{x \Delta}{10^{r-5}}.$$

Let a' a'' a''' &c. be the several $r - 5$ digits of x

then $x = a' 10^{r-6} + a'' 10^{r-7} +$ &c.

and $t \log N = r - 5 + t \log n + \Delta \left(\frac{a'}{10} + \frac{a''}{10^2} + \text{&c.} \right)$.

A small table of *proportional parts* or values of the term $\Delta \frac{a'}{0}$ for $x = 1$, or 2, or 3, up to $x = 9$ when x consists only of one digit, is given with some tables of logarithms, which greatly facilitates this calculation. For an example of the use of a table of proportional parts; See *Peacock's Algebra*, p. 565, Also "*Library of Useful Knowledge, Arithmetic and Algebra*," arts. 240; 242; in which last work, the student will find an excellent practical treatise on the nature and uses of logarithms.

(*c*) The following is an example of the application of logarithms, which from the principle it involves may properly find a place here.

To find the state of population in a country after any time, under given circumstances of births and mortality.

Let $P =$ population at any period

$\frac{1}{m} P =$ the number that die in one year,

$\frac{1}{b} P =$ the number of births in the same time.

Then $P \left(1 + \frac{1}{b} - \frac{1}{m}\right) = P \left(1 + \frac{m - b}{m b}\right)$

will be the state of the population at the end of one year; suppose this $= P_1$ then

$$P_1\left(1+\frac{m-b}{m\,b}\right)=P\left(1+\frac{m-b}{m\,b}\right)^2$$

will be the state of the population at the end of two years.

Similarly, after n years, the state

$$(A) \text{ will } = P\left(1+\frac{m-b}{m\,b}\right)^n;$$

$$\text{Hence, } \log A = \log P + n\log\left(1+\frac{m-b}{m\,b}\right)$$

$$= \log P + n\log\left(1+\frac{1}{p}\right);$$

putting $\frac{1}{p}$ for the fraction $\frac{m-b}{m\,b}$, which is always known under given circumstances of births and mortality; and any three of the quantities, $A, P, n,$ or $\frac{1}{p}$ being given the fourth may be found.

For some numerical examples in this theory, See *Bridge's Algebra*, p. 220.

CHAPTER XIV.

ON THE CALCULATION OF INTEREST AND OF THE VALUE OF ANNUITIES.

DEF. 1. *Interest*, is *the value* of the use of money; and this value is defined by a certain sum (called the *rate per cent*) which is paid for the use of every £100. for a certain definite time; generally one year.

DEF. 2. The whole sum which bears interest is called *the principal.*

DEF. 3. *Simple interest* is the interest of the principal alone.

DEF. 4. *Compound Interest* is the interest of the principal, together with that of the interest added to the principal as soon as it becomes due.

DEF. 5. *The Amount* after any time, is the principal added to the interest that has accrued during that time; it is therefore the whole sum then due.

203. The following notation is employed to denote these quantities. Simple interest (i); compound interest (I); principal (P); amount at simple interest (m); amount at compound interest (M); interest of £1. for one year (r); time during which the interest accrues (t).

204. *To find the amount of a given sum in any time at simple interest.*

Now, interest of £1. for one year $= r$;

$\therefore$ interest of £P for one year $= Pr$;

..... two years $= P \cdot 2\ r$

&c. &c. $=$ &c.

And generally interest of £P for t years $= P\,r\,t = i$;

$\therefore\ m = i + P$ (Def. 5.) $= P\,r\,t + P = P(1 + r\,t)$.

(*a*). Hence any three of the quantities m, r, t, or P being given, the fourth may be found from the equation

$$m = P(1 + r\,t).$$

(*b*). Since r is the interest of £1. for one year;

$\therefore\ 100 \cdot r =$ interest of £100. for one year $=$ rate per cent. (Def. 1.)

$$\therefore\ r = \frac{\text{rate per cent}}{100}.$$

205. *To find the amount of a given sum at compound interest.*

Let $R = 1 + r =$ £1. together with its interest for one year.

Then, amount of £P at the end of one year $= P(1 + r)$ Art. 202. This amount then becomes the principal upon which interest is to be calculated for the ensuing year, (Def. 4); so that writing $P(1 + r)$ for P, we shall have, amount at the end of the second year $= P(1 + r)^2$; and generally at the end of the t^{th} year;

$M = P(1 + r)^t = PR^t$, where $R = 1 + r =$ £1. together with its interest for one year.

(*a*). Any unit of time, as a day, a month, &c. may be selected as well as a year for the periodic times, at the end of each of which interest is to become due, and the reasoning will be the same as before, so that t may signify either a certain number of

years, or a certain number of parts of a year, the interest of £1, for the given unit of time being, in that case substituted for r.

(*b*). Hence if compound interest be calculated for every $\frac{1}{n^{\text{th}}}$ part of a year, r retaining the same value as in Art. 203, we shall have the amount at the end of the year, or,

$$M = P\left(1 + \frac{r}{n}\right)^n$$

$$= P\left(1 + r + \frac{n(n-1)}{1.2} \cdot \frac{r^2}{n^2} + \frac{n(n-1)(n-2)}{1.2.3} \cdot \frac{r^3}{n^3} + \&c.\right)$$

If n be infinite, or compound interest be allowed for every indefinitely small portion of time this series becomes

$$P\left(1 + r + \frac{r^2}{1.2} + \frac{r^3}{1.2.3} + \&c.\right) = Pe^t \text{ (Art. 194 (}a\text{).)}$$

(*c*). If the amount be required for t years and the $\frac{1}{n}^{\text{th}}$ part of a year; we have, amount at the end of t years $= PR^t = P_1$ suppose; then if x be the amount of P_1 for the $\frac{1}{n}$th of a year that is, the whole amount required, we shall have,

$$x = P_1 (1 + r)^{\frac{1}{n}} \text{ by } (a),$$

as r remains here unaltered; because each periodic time for paying the interest is still supposed to be one year, and not any fractional part of a year as in (*b*)

$$\therefore x = P_1 R^{\frac{1}{n}} = PR^t R^{\frac{1}{n}} = PR^{\frac{nt+1}{n}}.$$

(*d*) From the equation, $M = PR^t$, we get,

$$\log M = \log P + t \cdot \log R,$$

whence, any three of the quantities M, P, t, R, being given, the fourth may be found.

Def. 6. *Discount* is an allowance made for the payment of money before it becomes due.

206. Hence the sum paid *immediately*, ought, when improved by being placed out to interest in the meantime, to be equal to the sum (A) due at the appointed future time (t), and the difference between these two sums is the discount.

(1). *Discount at Simple Interest.*

Let a = sum paid immediately, then its *improved value* at the end of the time $t = a(1 + rt)$, Art. 201.

$$\therefore a(1 + rt) = A; \text{ and } a = \frac{A}{1 + rt};$$

$$\text{hence, discount} = A - a = A - \frac{A}{1 + rt} = \frac{Art}{1 + rt}.$$

(*a*). Hence a, or $\frac{A}{1 + rt}$, is the *present value* of a sum A due at the end of the time t.

(*b*). In practice the *interest* of the sum A for the time t is generally taken as its discount for the same time, this gives an advantage to the person paying, as he deducts the interest of A, the sum due at a *future* time, instead of the interest of $\frac{A}{1 + rt}$, its *present value,* but on the other hand the person receiving is freed from all uncertainty, or risk of a bad debt.

(2). *Discount at Compound Interest.*

Here, improved value of $a = aR^t = A$;

$$\therefore a = \frac{A}{R^t} \text{ and discount} = A - a = A\left(1 - \frac{1}{R^t}\right).$$

In this case, as before, a or $\frac{A}{R^t}$ is the *present value* of a sum A due at the time t.

(*c*). If compound interest be substituted for discount we have its value = amount − original sum

$$= AR^t - A = A(R^t - 1).$$

EQUATION OF PAYMENTS.

207. A *owes* B *the sum of* £P. *due at the end of the time* t, *and* £P′ *due at the end of the time* t′; *to find at what time the whole sum* P + P′ *must be paid at once, so that neither may be a loser.*

(1). *Simple Interest.*

Let x be the time required, then Interest of £P for time $x - t$ = discount of £ P' for time $t' - x$;

$$\text{or (by last Art.) } P\,r\,(x-t) = \frac{P'\,r\,(t'-x)}{1 + r\,(t'-x)},$$

which is a quadratic equation for determining x.

(*a*). In practice, interest is substituted for discount, in which case we have, interest of P for time $\overline{x - t}$ = interest of P' for time $\overline{t' - x}$

$$\text{or, } P\,r\,(x-t) = P'\,r\,(t'-x)$$

$$\text{whence } x = \frac{P\,t + P'\,t'}{P + P'}.$$

(*b*). Similarly, if a third sum (P''); a fourth (P''')&c. were due at the times t'', t''' &c. the equated time of payment for the whole sum would be

$$\frac{P\,t + P'\,t' + P''\,t'' + P'''\,t''' + \&c.}{P + P' + P'' + P''' + \&c.}$$

(2). *Equation of Payments at compound interest.*

Here, interest of P for time $x - t = P\,(R^{x-t} - 1)$ Art. 204 (*c*).

$$\text{Discount of } P' \ldots\ldots t' - x = P'\left(1 - \frac{1}{R^{t'-x}}\right);$$

$$\therefore P\,(R^{x-t} - 1) = P'\left(1 - \frac{1}{R^{t'-x}}\right) = P'\,\frac{R^{t'-x} - 1}{R^{t'-x}}$$

$$\text{or } P\,(R^{t'-t} - R^{t'-x}) = P'\,(R^{t'-x'} - 1)$$

that is, $P' + PR^{t'-t} = (P + P')\,R^{t'-s}$

whence, taking the logarithms and transposing,

$$x = \frac{\log(P + P') + t' \log R - \log(P' + PR^{t-t'})}{\log R}.$$

$$= t' + \frac{\log(P + P') - \log(M' + P')}{\log R}$$

where $M' = PR^{t-t'}$ = amount of P for time $t - t'$.

ON ANNUITIES.

208. *To find the amount of an annuity* (A) *left unpaid for* t *years.*

(1) *Simple Interest.*

1st Annuity bears interest for $t - 1$ years, and its amount

$= A(1 + r.\overline{t-1})$

2nd .....$t - 2$ ..... $= A(1 + r.\overline{t-2})$

3rd .....$t - 3$ ..... $= A(1 + r.\overline{t-3})$

.....

$\overline{t-1}^{\text{th}}$ ....1 ..... $= A(1 + r)$

t^{th} .....0 ..... $= A$;

$\therefore$ whole amount $= tA + Ar(1 + 2 + 3 \ldots\ldots \overline{t-1})$

$$= A\left(t + rt.\frac{t-1}{2}\right)$$

(2) *Compound Interest.*

1st Annuity bears interest for $t-1$ years and its amount $= AR^{t-1}$

2nd ..... $t-2$ .. .. $= AR^{t-2}$

.....

$\overline{t-1}^{\text{th}}$ ..... 1 ..... $= AR$

t^{th} ..... 0 ..... $= A$;

$\therefore$ whole amount $= A(1 + R + R^2 \ldots\ldots R^{t-1})$

$$= A.\frac{Rt - 1}{R - 1}$$

by the summation of a Geometric Progression.

209. *To find the present value of an annuity to continue for* t *years.*

(1) *Simple Interest.*

Present value of 1st annuity $= A\frac{1}{1+r}$ Art 203. (*a*).

$$\text{2nd} \ldots\ldots = A\frac{1}{1+2r}$$

$$\ldots\ldots$$

$$t^{\text{th}} \ldots\ldots = A\frac{1}{1+tr}.$$

$\therefore$ present value of whole annuity

$$= A\left(\frac{1}{1+r} + \frac{1}{1+2r} + \text{\&c.} + \frac{1}{1+tr}\right).$$

(*a*) The present value of an annuity estimated at simple interest, is sometimes defined to be, that sum, the improved value of which, would in the given number of years, be equal to the amount of the annuity; if P' be the present value on this hypothesis we have; Art. 206, improved value of $P' = P'(1+tr)$

$$= A\left(t + rt\frac{t-1}{2}\right) \therefore P = A\frac{t + rt\frac{t-1}{2}}{1+rt}.$$

but this is not the present value of the *annuity*, but the present value of its *amount*. See "*Wood's Algebra*," Art. 406. Also the treatise on Algebra in the "*Library of Useful Knowledge*," *Art.* 308.

(2) *Compound Interest.*

Present value of 1st annuity $= \frac{A}{R}$ Art. 204. (2).

$\ldots\ldots$ 2nd $\ldots\ldots = \frac{A}{R^2}$

$\ldots\ldots$ $\ldots\ldots$ $\ldots\ldots$

$\ldots\ldots$ t^{th}. $\ldots\ldots = \frac{A}{R^t}$

$\therefore$ present value of whole annuity $= A\left(\frac{1}{R} + \frac{1}{R^2} \cdot \ldots \cdot + \frac{1}{R^t}\right)$

which is a geometric progression of which the first term is $\frac{1}{R}$, the number of terms t, and the common ratio $\frac{1}{R}$, hence

$$\text{present value} = A \cdot \frac{\frac{1}{R}\cdot\frac{1}{R^t} - \frac{1}{R}}{\frac{1}{R} - 1} = A\frac{\frac{1}{R^t} - 1}{1 - R} = A\,\frac{1 - \frac{1}{R^t}}{R - 1}.$$

(*b*) To find the present value (P') of *the amount* of the annuity; we have, improved value of P' = amount of annuity

$$\text{or, } P'R^t = A\,\frac{R^t - 1}{R - 1};$$

$$\therefore P' = A.\,\frac{1 - \frac{1}{R^t}}{R - 1}$$

which agrees with the present value of the annuity found above; as it manifestly ought; although when calculated at simple interest these conclusions differ.

(*c*) Any three of the quantities P', A, R, t, being given, the fourth may be found from the equation

$$P' = A\,\frac{1 - \frac{1}{R^t}}{R - 1}.$$

(*d*) The present value of an annuity to commence at the end of t_1 years, and to continue for t_2 years = present value of an annuity for $t_1 + t_2$ years − present value of an annuity for t_2 years; which may be calculated on the preceding principles.

(*e*) If an annuity (A) be payable in n instalments during a year, the amount of such an annuity will be the same as the amount of an annuity, each of whose payments is $\frac{A}{n}$, the rate of interest $\frac{r}{h}$, and the number of years $n\,t$, hence

$$\text{amount} = \frac{A}{n} \frac{\left(1+\frac{r}{n}\right)^{nt} - 1}{\left(1+\frac{r}{n}\right) - 1} \quad \text{Art. 205. (2)}$$

$$= \frac{A}{r}\left\{\left(1+\frac{r}{n}\right)^{nt} - 1\right\}$$

$$\text{also present value} = \frac{A}{n} \frac{1 - \frac{1}{(1+\frac{r}{n})^{nt}}}{\frac{r}{n}} \quad \text{Art. 209. (2)}$$

$$= \frac{A}{r}\left\{1 - \frac{1}{\left(1+\frac{r}{n}\right)^{nt}}\right\}.$$

CHAP. XVII. SECTION I.

THE GENERAL THEORY OF SERIES.

ON RECURRING SERIES.

210. *To develope in a series involving ascending powers of* x *the fraction* $\frac{a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_{n-1} x^{n-1}}{1_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_n x^n}$;

$$\text{Assume, } \frac{a_0 + a_1 x + \&c. + a_{n-1} x^{n-1}}{1_0 + b_1 x + \&c. + b_n x^n}$$
$$= A_0 + A_1 x + A_2 x^2 + \ldots\ldots + A_n x^n \ldots . (a),$$

where the co-efficients A_0 , A_1 &c. remain to be determined; then by multiplying by $1_0 + b_1 x +$ &c. we have,

$$\begin{aligned} & a_0 + a_1 x + \&c. + a_{n-1} x^{n-1} \\ = {} & A_0 + A_1 x + A_2 x^2 + \&c. + A_n x^n \\ & + b_1 (A_0 x + A_1 x^2 + A_2 x^3 + \&c. + A_{n-1} x^n) \\ & \qquad + b_2 (A_0 x^2 + A_1 x^3 + \&c. + A_{n-2} x^n + \&c.) \\ & \qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \&c. \\ & \qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots b_{n-1} (A_1 x^n + \&c.) \\ & \qquad\qquad + b_n A_0 x^n + \&c. \end{aligned}$$

whence we have, by equating the co-efficients involving the same power of x (Art. 183)

$$A_0 = a_0$$
$$A_1 + b_1 A_0 = a_1 ; \text{ whence, } A_1 = a_1 - b_1 A_0 = a_1 - b_1 a_0 ;$$

and generally, we have from the co-efficient of x^n

$$0 = A_n + b_1 A_{n-1} + b_2 A_{n-2} + \&c. + b_{n-1} A_1 + b_n A_0 \ldots\ldots\ldots (a);$$

which is an equation giving the value of any co-efficient A_n, in terms of the preceding co-efficients A_{n-1}, A_{n-2}, &c.

(a). If we put u_n, u_{n-1}, &c. for the *terms* of the series, we shall have

$$u_n = A_n x^n; \; u_{n-1} = A_{n-1} x^{n-1}; \; \&c.$$

whence, multiplying equation (a) by x^n and transposing, we obtain,

$$u_n = - b_1 u_{n-1} x - b_2 u_{n-2} x^2 - \&c. - b_n u_0 x^n.$$

(b). Series, similar to the above, in which any term as u_n, may be deduced from those which precede it, by multiplying those terms by a certain invariable quantity are denominated "*Recurring Series.*" The invariable quantity, as

$$- b_1 - b_2 - \&c. - b_n$$

in the above example, being called "*the scale of relation.*"

211. *Every recurring series may be resolved into a certain number of geometric series.*

For every recurring series arises from several fractions having a common denominator of the form

$$b_0 + b_1 x + b_2 x^2 + \&c. + b_n x^n$$

the numerators being a_0; $a_1 x$; $a_2 x^2$; &c. Art. 210 (b); and each of these fractions may again be resolved into others of the form

$$\frac{L}{1 + \alpha x}, \frac{M}{1 + \beta x} \; \&c. \text{ or } \frac{k}{(1 + j x)^n} \; \&c.$$

each of which last is the generating fraction of a geometric progression, Art. 121 (a); such a term as $\frac{k}{(1 + j x)^n}$ being the *product* of n geometric progressions.

212. To find the general term of a recurring series, we must resolve it into its component geometric series, and find the general term in each.

Ex. Given the series,

$$1 + x + 2x^2 + \frac{3}{2}x^3 + \frac{9}{4}x^4 + \&c. = \frac{2 + x + x^2}{2 - x - 2x^2 + x^3}$$

the scale of relation being $-\frac{1}{2}x^3 + x^2 + \frac{1}{2}x$;

Now, $\frac{2 + x + x^2}{2 - x - 2x^2 + x^3} = \frac{2}{1 - x} + \frac{\frac{1}{3}}{1 + x} - \frac{\frac{8}{3}}{2 - x}$ see Art. 184;

each of which fractions generates a geometric progression, the sum of whose general terms is

$$\frac{1}{3}x^n \left\{6 + (-1)^n - \left(\frac{1}{2}\right)^{n-2}\right\}$$

which is therefore the general term of the series.

213. The sum of a recurring series may be found by taking separately the sum of its component geometric progressions, and then adding those sums.

214. *Having given a recurring series, to find the geometric progressions of which it is composed.*

Let the unknown ratios of these progressions be r_1, r_2, r_3, &c. and their first terms a_1, a_2, a_3 &c. then the co-efficient of the general term of the recurring series is

$$A_n = a_1 r_1^n + a_2 r_2^n + a_3 r_3^n + \&c.$$

and $A_n = -b_1 A_{n-1} - b_2 A_{n-2} - \&c. - b_n A_0$; Art. 210.

Now if the general co-efficient, A_n, consisted only of one quantity as $a_1 r_1^n$, we should have, $A_{n-1} = a_1 r_1^{n-1}$; &c = &c. and we should have by substituting in (1) and suppressing the common factor,

$$a_1 (r_1{}^n + b_1 r_1{}^{n-1} + b_2 r_1{}^{n-2} + \&c.) = 0 \ldots . (2)$$

an equation which would result immediately from the given scale of relation; the co-efficients of $r_1{}^{n-1}$, $r_1{}^{n-2}$ &c. being those of the scale of relation.

Similarly, if the general term consisted of two quantities, $a_1 r_1{}^n + a_2 r_2{}^n$; we should have,

$$a_1 (r_1{}^n + b_1 r_1{}^{n-1} + \&c.) + a_2 (r_2{}^n + b_1 r_2{}^{n-1} + \&c.) = 0;$$

an equation which would evidently be satisfied by assuming for r_1 and r_2 two of the roots of equation (2), whatever be the values of a_1, a_2, &c.

The same reasoning will apply when the general term of the series consists of n terms, as in equation (1); in which case the n roots of equation (2) would satisfy the resulting equation,

$$a_1 (r_1{}^n + \&c.) + a_2 (r_2{}^n + \&c.) + a_3 (r_3{}^n + \&c.) + \&c. = 0,$$

when put for r_1, r_2, r_3, &c. respectively.

Hence the ratios of the progressions required will be found by finding the n roots of equation (2).

In order to find the quantities a_1, a_2, &c. we have, generally,

$$A_n = a_1 r_1{}^{n-1} + a_2 r_2{}^{n-1} + a_3 r_3{}^{n-1} + \&c.;$$

whence,
$$\left.\begin{aligned} A_1 &= a_1 + a_2 + a_3 + \&c. \\ A_2 &= a_1 r_1 + a_2 r_2 + a_3 r_3 + \&c. \\ \&c. &= \&c. \end{aligned}\right\}$$

and the quantities A_1, A_2, &c. are supposed to be known from the given first terms of the proposed series; hence a_1, a_2, &c. are known by elimination.

Ex. Given the series

$$1 + \frac{5}{2} x - \frac{7}{4} x^2 + \&c.$$

whose scale of relation is $-\frac{7}{2} x^2 + \frac{5}{2} x$, to find the general term.

Here equation (2) becomes $r_1^2 + \frac{1}{2}r - \frac{1}{2} = 0$; whose roots are $\frac{1}{2}$ and -1, which are therefore the ratios of the progressions required: also we have,

$$A_1 = -1 = a_1 + a_2; \text{ and, } A_2 = \frac{5}{2} = \frac{1}{2}a_1 - a_2;$$

from which two equations

$$a_1 = 1; \text{ and } a_2 = -2; \text{ and } A_n = \left(\frac{1}{2}\right)^{n-1} - 2(-1)^{n-1}.$$

See *Francœur*, Art. 583.

CHAP. XVII. SECTION II.

THE DIFFERENTIAL METHOD OF SERIES.

215. If u_1, u_2, &c. u_n, be the first, second, &c. n^{th}, terms of a series, then the quantities $u_2 - u_1$; $u_3 - u_2$; $u_n - u_{n-1}$, &c. are called the *differences* of the quantities u_1, u_2, &c. u_{n-1}; and are thus denoted,

$$u_n - u_{n-1} = \Delta u_{n-1};\ u_2 - u_1 = \Delta u_1;\ \&c.$$

216. The differences of these differences may also be taken which would then be denoted thus,

$$\Delta^2 u_n = \Delta u_{n+1} - \Delta u_n;\ \&c.$$

217. *To find a general expression for the* n$^{\text{th}}$ *term of any series.*

We have $u_2 = u_1 + \Delta u_1$; Art. 215;

also, $u_3 = u_2 + \Delta u_2$

substitute in this the value of u_2 given in the former line;

$$\therefore u_3 = u_1 + \Delta u_1 + \Delta (u_1 + \Delta u_1)$$
$$= u_1 + 2\Delta u_1 + \Delta^2 u_1.$$

Similarly, $u_4 = u_2 + 2\Delta u_2 + \Delta^2 u_2$;

substitute again the value of u_2;

$$\therefore u_4 = u_1 + \Delta u_1 + 2\Delta (u_1 + \Delta u_1) + \Delta^2 (u_1 + \Delta u_1)$$
$$= u_1 + 3\Delta u_1 + 3\Delta^2 u_1 + \Delta^3 u_1,$$

where it is evident the co-efficients will follow the law of the binomial theorem; hence, generally,

$$u_{n+1} = u_1 + n\,\Delta u_1 + \frac{n\,(n-1)}{1\,.\,2}\,\Delta^2 u_1 + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,\Delta^3 u_1 + \&c.$$

and writing $n - 1$ for n we shall have,

$$u_n = u_1 + (n-1)\,\Delta u_1 + \frac{(n-1)\,(n-2)}{1\,.\,2}\,\Delta^2 u_1 + \&c.$$

Ex. Required the n^{th} term of the series, 1, 3, 5, 7, &c.

Here, $u_1 = 1$; $\Delta u_1 = 3 - 1 = 2$; $\Delta^2 u_1 = 0$;

$\therefore\ u_n = 1 + (n-1)\,2 = 2n - 1.$

218. *To find the sum of any series*, u_1, u_2, u_3, &c.

First, suppose the sum of the following series were required;

$0,\ u_1,\ u_1 + u_2,\ u_1 + u_2 + u_3,\ u_1 + u_2 + u_3 + u_4,$ &c.

Then putting u_0 for 0, and $n + 1$ for n in the last article, observing that u_0 is the first term of the series, we find the $\overline{n+1}^{\text{th}}$ term of this series

$$= u_0 + n\,\Delta u_0 + \frac{n\,(n-1)}{1\,.\,2}\,\Delta^2 u_0 + \&c.$$

that is, $u_1 + u_2 + u_3 +$ &c. to n terms

$$= u_0 + n\,\Delta u_0 + \frac{n\,(n-1)}{1\,.\,2}\,\Delta^2 u_0 + \&c.$$

Now $u_0 = 0$; $\Delta u_0 = u_1 - u_0 = u_1$; $\Delta^2 u_0 = \Delta u_1$; &c.

$\therefore\ u_1 + u_2 +$ &c. to n terms

$$= n u_1 + \frac{n\,(n-1)}{1\,.\,2}\,\Delta u_1 + \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}\,\Delta^2 u_1 + \&c.$$

which gives the sum, not of the series with which we originally started, but of the series,

$u_1,\ u_2,\ u_3,$ &c. to n terms.

Ex. *Required the sum of* $1^3 + 2^3 + 3^3 + 4^3 +$ &c. *to* n *terms.*

1 , 8 , 27 , 64 , 125 ; $\therefore u_1 = 1.$

1st differences are 7 , 19 , 37 , 61,..... $\Delta u_1 = 7.$

2nd..... 12 , 18 , 24,..... $\Delta^2 u_1 = 12.$

3rd..... 6 , 6, ... $\Delta^3 u_1 = 6.$

4th.....0..... $\Delta^4 u_1 = 0.$

$$\therefore \text{sum required} = n + \frac{n(n-1)}{1 \cdot 2} \cdot 7 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot 12$$

$$+ \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 6$$

$$= \frac{n^4 + 2n^3 + n^2}{4} = \left(\frac{n(n+1)}{2}\right)^2.$$

CHAP. XVII. SECTION III.

ON THE METHOD OF INCREMENTS.

219. ANY variable quantity, as the sum of a series to n terms, where n is variable and assignable at pleasure, is called *an integral.*

The magnitude by which it increases at one step is called the increment of the integral.

220. *To find the increment of an integral, which consists of a number of factors in Arithmetic Progression.*

Let the integral be $S_n = r_1\, r_2\, r_3\, r_4 \ldots\ldots r_{n-1}$ where

$r_1 = a + b$; $r_2 = a + 2b$; $r_{n-1} = a + \overline{n-1}\, b$; &c.

The succeeding value of this integral will be $r_2\, r_3 \ldots\ldots r_n$ the increment is therefore

$$\begin{aligned}\Delta S_n = S_{n+1} - S_n &= r_2\, r_3 \ldots\ldots r_n - r_1\, r_2 \ldots\ldots r_{n-1} \\ &= r_2\, r_3 \ldots\ldots r_{n-1}\ (r_n - r_1) \\ &= r_2\, r_3 \ldots\ldots r_{n-1}\ (a + n\,b - \overline{a + b}) \\ &= r_2\, r_3 \ldots\ldots r_{n-1}\ (\overline{n-1}\,.\,b)\end{aligned}$$

(*a*) If $S_n = r_1\, r_2 \ldots\ldots r_{n-1} + C$, where C is an invariable quantity, then we find in the same way,

$$\Delta S_n = r_2\, r_3 \ldots\ldots r_{n-1}\ (\overline{n-1}\,.\,b)$$

so that this latter increment corresponds both to

$$r_1\, r_2 \ldots\ldots r_{n-1} \text{ and to } r_1\, r_2 \ldots\ldots r_{n-1} + C.$$

(*b*) Conversely, if we have given an increment of the form $r_2 r_3 \ldots . r_{n-1}$ we may conclude that its integral is

$$\frac{r_1 r_2 r_3 \ldots . r_{n-1}}{(n-1)\,b} + C$$

where C is to be determined by the nature of the question.

Hence the rule for finding the integral of an increment which consists of factors in Arithmetic Progression; "multiply it by the preceding term in the progression, and divide by the number of terms thus increased, and by the common difference."

Ex. To sum to n terms the series $1^2 + 2^2 + 3^2 +$ &c.

Here the increment of the sum

$$= (n+1)^2 = n\,(n+1) + n + 1,$$

hence the integral or sum is,

$$\frac{(n-1)\,n\,(n+1)}{3} + \frac{n\,(n+1)}{2} + C; \text{ let } n = 1,$$

$$\therefore \text{ sum} = \frac{1 \cdot 2}{2} + C, \text{ but in that case; Sum} = 1;$$

$$\therefore C = 0; \text{ and the sum of the series to } n \text{ terms}$$

$$= \frac{(n-1)\,n\,(n+1)}{3} + \frac{n\,(n+1)}{2} = \frac{(2n+1)\,n\,(n+1)}{2 \cdot 3}$$

Def. *Figurate numbers* are formed by making the n^{th} term of any order equal to the sum of the first n terms of the preceding order; thus in the annexed table, 35, which is the 5th term of the 4th order = first 5 terms of the 4th order

$$= 1 + 3 + 6 + 10 + 15$$

1st order,	1, 1, 1, 1, 1
2nd	1, 2, 3, 4, 5
3rd	1, 3, 6, 10, 15
4th	1, 4, 10, 20, 35
&c.	&c.

221. *To find the sum of* n *terms of the* mth *order of figurate numbers.*

Sum of n terms of the 1st order $= n$; $\therefore n + 1 =$ increment of the sum of n terms of the second order.

Hence the integral or sum of n terms of the 2nd order

$$= \frac{n(n+1)}{2}.$$

$\therefore \frac{(n+1)(n+2)}{2} =$ increment of sum of n terms of the 3rd order

$\therefore$ Sum of n terms of the 3rd order $= \frac{n(n+1)(n+2)}{2.3}$.

Hence generally, $\frac{n(n+1)(n+2)\ldots\ldots(n+m-1)}{1.2.3\ldots\ldots m} =$ sum of n terms of the m^{th} order.

222. *To find the increment of* $\frac{1}{r_1 r_2 r_3 \ldots\ldots r_{n-1}}$ *the quantities* $r_1 r_2$ &c. *representing as before (Art.* 220*) terms of an Arithmetic progression,*

$$\text{Increment} = \frac{1}{r_2 r_3 r_4 \ldots\ldots r_n} - \frac{1}{r_1 r_2 r_3 \ldots\ldots r_{n-1}}$$

$$= \frac{1}{r_1 r_2 \ldots\ldots r_n}(r_1 - r_n)$$

$$= \frac{1}{r_1 r_2 \ldots\ldots r_n}(a+b-\overline{a+nb}) = \frac{-\overline{n-1}\,b}{r_1 r_2 \ldots\ldots r_n}.$$

(*a*) Hence if we have an increment $= \frac{1}{r_1 r_2 r_3 \ldots\ldots r_n}$; its integral will $= \frac{1}{-\overline{n-1}\,b.r_1 r_2 \ldots r_{n-1}} + C$; and we have this rule to integrate the reciprocal of a quantity which consists of factors in Arithmetic Progression; "reject the last term in

the denominator, and divide by the number of terms left, and also by the common difference taken negatively."

Ex. To sum the series $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7}$ + &c.

Here the general term $= \frac{1}{(2n-1)(2n+1)}$;

And the increment of the sum $= \frac{1}{(2n+1)(2n+3)}$;

The integral of which $= \frac{1}{(2n+1) \times -2} + C$, since the common difference is in this case 2;

Let $n = 1$; $\therefore \frac{1}{1 \cdot 3} = \frac{1}{-6} + C$; $\therefore C = \frac{1}{2}$

and sum required $= \frac{1}{2} - \frac{1}{4n+2}$.

(*a*) Let $n = \infty$, then sum *ad infinitum* $= \frac{1}{2}$.

CHAP. XVII. SECTION IV.

ON THE LIMITS OF SERIES.

223. *If* $u_0 + u_1 x + u_2 x^2 \ldots\ldots + u_n x^n + \&c.$ *be a series and* R *the* greatest *inverse ratio of any two of its consecutive co-efficients, then will each of its terms be less than, or equal to the corresponding terms of the geometric series*

$$u_0 (1 + R x + R^2 x^2 + \ldots\ldots + R^n x^n + \&c.)$$

For let $R = \frac{u_p}{u_{p-1}}$ be the *greatest* inverse ratio of two consecutive co-efficients, u_p, u_{p-1}, and let $\frac{u_{p-m}}{u_{p-m-1}}$ be the inverse ratio of any two other co-efficients.

Then by hypothesis $\frac{u_1}{u_0} \angle R$;

$$\frac{u_2}{u_1} \angle R;$$

&c. $\angle$ &c.; until we come to,

$$\frac{u_{p-m}}{u_{p-m-1}} \angle R;$$

and by multiplying these inequalities together, we have,

$$\frac{u_1 u_2 \ldots\ldots u_{p-m}}{u_0 u_1 \ldots\ldots u_{p-m-1}} \angle R^{p-m}; \text{ or, } \frac{u_{p-m}}{u_0} \angle R^{p-m};$$

that is, the term u_{p-m} is less than the corresponding term, $u_0 R^{p-m}$, of the geometric series.

Similarly, by multiplying together the factors

$$\frac{u_1}{u_0} \angle R;\ \frac{u_2}{u_1} \angle R;\ \text{&c. up to, } \frac{u_p}{u_{p-1}} = R;\ \text{and}$$

$$\frac{u_{p+1}}{u_p} \angle R;\ \text{&c. up to } \frac{u_{p+m}}{u_{p+m-1}} \angle R;$$

we may prove that u_{p+m} is $\angle\ u_0 R^{p+m}$.

$$(a).\quad \text{If } r = \frac{u_1}{u_0} = \frac{u_2}{u_1} = \text{&c. up to } \frac{u_m}{u_{m-1}}.$$

$$\text{Then } r^m = \frac{u_1 u_2 \ldots u_m}{u_0 u_1 \ldots u_{m-1}} = \frac{u^m}{u_0}$$

and if this hold until $m = n$; in that case the corresponding terms of the two series are equal as far as the n^{th} term.

224. If r be the *least* inverse ratio of any two consecutive co-efficients of the series

$$u_0 + u_1 x + u_2 x^2 \ldots\ldots + u_n x^n + \text{&c.}$$

we may shew by a proof precisely similar to that in the last Art., that each of the terms of the series are severally equal to or *greater* than the corresponding term of the series

$$u_0 + u_0 r x + u_0 r^2 x^2 \ldots + u_0 r^n x^n + \text{&c.}$$

Hence the three following series are arranged in the order of their magnitude;

$$u_0 + u^0 R x + u_0 R^2 x^2 \ldots + u_0 R^n x^n + \text{&c. } ad\ infinitum \left(= \frac{u_0}{1 - R x}\right)$$

$$u_0 + u_1 x + u_2 x^2 \ldots + u_n x^n + \text{&c. } ad\ infinitum (= S. \text{ suppose};)$$

$$u_0 + u_0 r x + u_0 r^2 x^2 \ldots + u_0 r^n x^n + \text{&c. } ad\ infinitum \left(= \frac{u_0}{1 - r x}\right)$$

and since S is intermediate in value between $\frac{u_0}{1 - R x}$ and $\frac{u_0}{1 - r x}$

(when $r\,x$ and $R\,x$ are less than 1, Art. 121) it differs from either of them by a quantity less than

$$\frac{u_0}{1 - R\,x} - \frac{u_0}{1 - r\,x} = u_0 \frac{(R - r)\,x}{(1 - R\,x)\,(1 - r\,x)}.$$

(*a*). If the series $u_0 + u_1\,x +$ &c. do not begin to converge until after the p^{th} term, then if S_p be the sum of the first p terms we may find the limits of the whole series, by finding, as before, those of $u_{p+1}\,x^{p+1} + u_{p+2}\,x^{p+2} +$ &c. which will be

$$\frac{u_{p+1}}{1 - R\,x} \text{ and } \frac{u_{p+1}}{1 - r\,x}.$$

Ex. *To find the limiting series of* $(1 + x)^n$.

The inverse ratio of any two consecutive terms is

$$\frac{u_p}{u_{p-1}} = \frac{n - p + 1}{p}, = \frac{n + 1}{p} - 1;\ p \text{ being variable.}$$

Now this decreases as p increases, and *vice versa;* when $p = \infty$; then $\frac{u_p}{u_{p-1}} = -1$ which is therefore the value of r;

Again, the *least* value of p is when $p = 1$ in which case

$$\frac{u_p}{u_{p-1}} = \frac{n + 1}{1} - 1 = n$$

which is therefore the value of R;

Hence the series for $(1 + x)^n$ differs from either

$$\frac{1}{1 - n\,x} \text{ or } \frac{1}{1 + x}$$

by a quantity less than $\frac{(n + 1)x}{(1 - n\,x)(1 + x)}$.

225. A series becomes convergent as soon as any term $u_p\,x^p$ becomes $< u_{p-1}\,x^{p-1}$; if then, we equate these terms, the whole number equal to, or next greater than, p, will give the

term where the convergency commences; for the series for $(1 + x)^n$ we have $\left(\frac{n+1}{p} - 1\right) x = 1$; whence $p = \frac{n+1}{1 + \frac{1}{x}}$.

226. *To find the limits of the series*

$$u_{t+1}\, x^{t+1} + u_{t+2}\, x^{t+2} + \&c.$$

where u_{t+1} u_{t+2} &c. *are the* $\overline{t+1}\,|^{th}$ $\overline{t+2}\,|^{th}$ &c. *co-efficients of* $(1 - x)^n$.

Here the inverse ratio of any two consecutive co-efficients, as

$$\frac{u_p}{u_{p-1}} \text{ is } = \frac{n - p + 1}{p} = \frac{n+1}{p} - 1,$$

which (as before) is greatest when p is least; that is, when $p = t$; therefore $R = \frac{n+1}{t} - 1$; and $r = -1$;

Hence the sum of the series $u_{t+1}\, x^{t+1} + u_{t+2}\, x^{t+2} +$ &c. is included between the limits

$$\frac{u_{t+1}}{1 - r\,x} \text{ and } \frac{u_{t+1}}{1 - R\,x};$$

$$\text{or between, } \frac{u_{t+1}}{1+x} \text{ and } \frac{u_{t+1}}{1 + \left(1 - \frac{n+1}{t}\right)x}.$$

(*a*). Let S_t be the sum of the first t terms of the series for $(1 + x)^n$; then the above are the limits of $S_t + u_{t+1}\, x^{t+1} +$ &c. and therefore $u_{t+1} x^{t+1} +$ &c. differs from S_t by a quantity less than

$$\frac{u_{t+1}}{1 + \left(1 - \frac{n+1}{t}\right)x} - \frac{u_{t+1}}{1+x}$$

$$= u_{t+1} \frac{\frac{n+1}{t}\, x}{\left\{1 + \left(1 - \frac{n+1}{t}\right)x\right\}\left\{1 + x\right\}}$$

Ex. To find within what limits of error the square root of 5 is given by the first five terms of the series $2\sqrt{1 + \frac{1}{4}}$.

The sixth term, or $u_{t+1} = ,0000534$, nearly;

$\therefore$ the first five terms which are $= 2,236026$, exceed the $\sqrt{5}$ by a quantity less than

$$,0000534 \cdot \times \frac{\frac{\frac{1}{2}+1}{5} \times \frac{1}{4}}{\left\{1+\left(1-\frac{\frac{1}{2}+1}{5}\right)\frac{1}{4}\right\}\left\{1+\frac{1}{4}\right\}}.$$

See, *Peacock's Algebra*, p. 290; whence the above theory has been principally taken.

CHAPTER XVIII.

ON THE EXPANSION OF A MULTINOMIAL.

227. *To find the general term of*

$$(a + bx + cx^2 + dx^3 + \&c.)^n$$

Now, $e^{a+bx+cx^2+\&c.} = e^a \cdot e^{bx} \cdot e^{cx^2}$. &c. or expanding, Art. 191.

$$1 + \frac{(a + bx + cx^2 + \&c.)}{1} + \frac{(a + bx + cx^2 + \&c.)^2}{1 \cdot 2} \ldots\ldots$$

$$+ \frac{(a + bx + cx^2 + \&c.)^n}{1 \cdot 2 \cdot 3 \ldots\ldots n} + \&c.$$

$$= \left\{1 + \frac{a}{1} + \frac{a^2}{1 \cdot 2} + \&c.\right\}\left\{1 + \frac{bx}{1} + \frac{b^2x^2}{1 \cdot 2} + \&c.\right\}\left\{1 + \frac{cx^2}{1} + \frac{c^2x^4}{1 \cdot 2} + \&c.\right\} \&c.$$

The multiplication of the factors composing the last side of this equation will produce terms such as the following one

$$\frac{a^\alpha}{1 \cdot 2 \ldots . \alpha} \cdot \frac{(bx)^\beta}{1 \cdot 2 \ldots . \beta} \cdot \frac{(cx^2)^\gamma}{1 \cdot 2 \ldots . \gamma} \cdot \frac{(dx^3)^\delta}{1 \cdot 2 \ldots . \delta} \&c.;$$

and the terms of the same denomination on each side of the equation must be equal, hence, we shall have

$$\frac{(a + bx + cx^2 + dx^3 + \&c.)^n}{1 \cdot 2 \cdot 3 \ldots\ldots n}$$

equal to the sum of all such terms as that above; subject to the restriction that the whole index of each term $= n$; hence, the general term of

$(a + bx + cx^2 + \&c.)^n$ is $\frac{1.2.3 \ldots . n \times a^\alpha \cdot b^\beta \cdot c^\gamma \cdot \&c.}{1.2 \ldots \alpha \times 1.2 \ldots \beta \times 1.2 \ldots \gamma \times \&c.} x^{\beta + 2\gamma + 3\delta + \ldots\ldots}$

(*a*) If it were required to assign the co-efficient of x^p we should have $\beta + 2\gamma + 3\delta + \&c. = p$ and this equation must be satisfied in every possible way, in other words, the number p must be broken into parts, β, 2γ, 3δ, &c. (not exceeding n in number) in every possible way between 1 and the highest power of x which enters into the multinomial to be expanded; the numbers in these several *fractures*, will correspond to the several particular terms which make up the whole co-efficient of x^p.

(*b*) If the given multinomial were $(a + b + c + d + \&c.)^n$ its general term would be, simply

$$\frac{1.2.3 \ldots\ldots n \times a^\alpha b^\beta c^\gamma d^\delta \&c.}{1.2 \ldots\ldots \alpha \times 1.2 \ldots\ldots \beta \times 1.2 \ldots\ldots \gamma \times 1.2 \ldots\ldots \delta \&c.};$$

where, $\alpha + \beta + \gamma + \delta + \&c. = n$.

Ex. 1. To find the term of $(a + b + c)^{10}$ involving $a^5 b^3 c^2$.

Here $n = 10$; $\alpha = 5$; $\beta = 3$; $\lambda = 2$,
and the term required is therefore

$$\frac{1.2.3 \ldots\ldots 10.a^5 b^3 c^2}{1.2.3.4.5 \times 1.2.3 \times 1.2} = 2520\, a^5 b^3 c^2.$$

Ex. 2. To find the co-efficient of x^3 in the expansion of $(a + bx + cx^2 + dx^4 + \&c.)^n$.

Here we have $\beta + 2\gamma + 3\delta = 3$;

This equation may be satisfied in the three following ways;

$$\left.\begin{array}{l} \beta = 0;\ \gamma = 0;\ \delta = 1; \\ \beta = 1;\ \gamma = 1;\ \delta = 0; \\ \beta = 3;\ \gamma = 0;\ \delta = 0; \end{array}\right\}$$

And since the whole index of any term must $= n$; we have, corresponding to the last three sets of equations, the three following for the simultaneous values of α;

$$\left.\begin{array}{rl} \alpha + 1 = n; & \therefore\ \alpha = n - 1; \\ \alpha + 1 + 1 = n; & \therefore\ \alpha = n - 2; \\ \alpha + 3 = n; & \therefore\ \alpha = n - 3; \end{array}\right\}$$

Hence the term required is,

$$1.2.3....n\left\{\frac{a^{n-1}\,d}{1.2.3....(n-1)\times 1}+\frac{a^{n-2}\,b\,c}{1.2.3....(n-2)\times 1\times 1}\right.$$

$$\left.+\frac{a^{n-3}\,b^3}{1.2.3....(n-3)\times 1.2.3}\right\}x^3$$

$$=\left\{n\,a^{n-1}\,d+n(n-1)\,a^{n-2}\,b\,c+\frac{n(n-1)(n-2)}{1.2.3}\,a^{n-3}\,b^3\right\}x^3.$$

For more examples, see *Peacock's Algebra*, p. 301.

226. *To find the number of terms of the expanded polynomial* $(a_1 + a_2 + a_3 \ldots\ldots a^m)^n$.

The number of terms of $(a_1 + a_2)^n$ is $n + 1$; Art. 138 (a).

Again,

$$(a_1+a_2+a_3)^n=(a_1+a_2)^n+u_1(a_1+a_2)^{n-1}a_3+u_2(a_1+a_2)^{n-2}a_3^2$$
$$+\ldots\ldots\ldots+u_{n-1}(a_1+a_2)\,a_3^{n-1}+a_3^n$$

u_1, u_2, &c. being the co-efficients of the binomial theorem.

Now by the first case, number of terms in $(a_1+a_2)^n = n+1$

$$\ldots\ldots\ldots\ldots\ldots\ldots(a_1+a_2)^{n-1}=n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(a_1+a_2)^{n-2}=n-1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_3=1$$

Hence the whole number of terms in $(a_1 + a_2 + a_3)^n$ will $=$ the sum of the series $1, 2, \ldots\ldots n, \overline{n+1}, = \frac{(n+1)(n+2}{1.2}$.

Similarly,

$$(a_1+a^2+a_3+a_4)^n=(a_1+a_2+a_3)^n+u_1(a_1+a_2+a_3)^{n-1}a_4$$
$$+\ldots\ldots\ldots a_4^n$$

and, by the last case, number of terms in

$$(a_1 + a_2 + a_3)^n = \frac{(n+1)(n+2)}{1 \cdot 2}$$

$$(a_1 + a_2 + a_3)^{n-1} = \frac{n(n+1)}{1 \cdot 2}$$

$$(a_1 + a_2 + a_3)^{n-2} = \frac{(n-1)n}{1 \cdot 2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(a_1 + a_2 + a_3) = 3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots a_4^n = 1$$

Hence the whole number of terms in $(a_1 + a_2 + a_3 + a_4)^n$ will equal the sum of the series,

$$1, 3, \ldots\ldots \frac{n(n+1)}{1 \cdot 2}, \frac{(n+1)(n+2)}{1 \cdot 2} = \frac{(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3}.$$

Generally, when we have m terms in the polynomial, the process will evidently be the same as that of finding the $\overline{n+1}^{\text{th}}$ term of the m^{th} order of Figurate numbers; p. 159.

Hence, No. of terms in

$$(a_1 + a_2 + \ldots\ldots a_m)^n = \frac{(n+1)(n+2)\ldots\ldots(n+m-1)}{1 \cdot 2 \cdot 3 \ldots\ldots (m-1)}.$$

(*a*). From what has preceded, it is manifest that the last formula expresses the number of ways in which a number (n) may be broken into two, three, four, &c. or m parts between 0 and m inclusive.

Printed by J. Hall, Cambridge.